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A FURTHER NOTE ON THE ARRANGEMENT OF VARIETY TRIALS: QUASI-LATIN SQUARES

By F. YATES, M.A.
Rothamsted Experimental Station

I. Introduction

A new method of arranging agricultural field trials involving a large number of varieties has recently been described by the author (3) under the name of "pseudo-factorial arrangements", but better, perhaps, termed *quasi-factorial* arrangements. Such arrangements are likely to be of utility in any experimental work in which a large number of treatments have to be compared and in which the experimental material falls into small groups of closely similar units.

In a quasi-factorial varietal trial the varieties are divided into sets in two or more ways, the varieties of each set being arranged in one or more randomized blocks. The block size can thus be kept small even when the number of varieties is very large, and all use of controls is avoided. In a two-dimensional quasi-factorial arrangement of 81 varieties, for example, the varieties, after being numbered at random from 1 to 81, are divided into a group of 9 sets consisting of varieties 1–9, 10–18, 19–27, ..., 73–81, and a similar group of 9 sets consisting of varieties (1, 10, 19, ..., 73), (2, 11, 20, ..., 74), ..., (9, 18, 27, ..., 81), each of these sets being arranged at random in one or more blocks of 9 plots. It will be seen that these sets form the rows and columns of a diagrammatic square of the varietal numbers.

In a square two-dimensional quasi-factorial arrangement further divisions of the varieties into groups of sets are also possible, each group being such that every set of the group includes one and only one variety from each set of every other group. If in the case of p^2 varieties $p+1$ such groups are formed, a completely orthogonal system results. Since every two treatments then occur together once and once only in a block, we arrive at a special case of the type of arrangement described in (4) and there called an arrangement in symmetrical incomplete randomized blocks.

In the present paper a further extension of the quasi-factorial principle is described, whereby differences associated with two different groupings of the experimental material can be simultaneously eliminated. This type of arrangement may be called an arrangement in *quasi-Latin squares**, from the analogy with ordinary Latin square arrangements. In varietal trials in quasi-Latin squares each complete replication of the varieties is arranged in the field in a square pattern, all differences between both rows and columns being eliminated from the varietal comparisons, just as they are in an ordinary Latin square.

Quasi-Latin squares are less flexible than ordinary quasi-factorial arrangements, since

* Or alternatively an arrangement in *lattice squares* (see note on nomenclature at the end of the paper).

the number of varieties or treatments must be a perfect square (or a perfect cube) and certain perfect squares, in particular 36, are inadmissible. The designs are likely to be of considerable practical utility, however, because if the two sources of variation associated with two different groupings of the experimental material are of equal magnitude, the simultaneous elimination of both sources is more than twice as effective in reducing experimental error as is the elimination of one source only. The effectiveness of Latin square arrangements in agriculture, for example, has long been recognized.

II. Structure of quasi-Latin squares

If the number of varieties is a perfect square (equal to p^2 say), then for certain values of p it is possible to divide the varieties into $p+1$ orthogonal groups of p sets (each set containing p varieties), i.e. in such a manner that each set of any one group of sets contains one and only one variety from each set of any other group of sets. The three groups of sets corresponding to the rows, columns and letters of a Latin square fulfil the conditions of orthogonality. The $p+1$ groups can therefore be formed from a completely orthogonal set of $p-1$ squares. Such completely orthogonal sets are known to exist for values of p which are prime numbers, and also for $p=4$, 8 and 9 (1). No such set exists for $p=6$ (2).

If $p+1$ such groups of sets exist, then the p^2-1 degrees of freedom representing differences between varieties partition into $p+1$ groups of $p-1$ degrees of freedom, each group corresponding to the $p-1$ contrasts between the p sets of the corresponding group of sets. Each replication may therefore be arranged in the field in the form of a square of which the rows correspond to one group of sets and the columns to a second, so that in every replication the degrees of freedom corresponding to two groups of sets will be confounded with row or column differences. If p is odd and there are $\frac{1}{2}(p+1)$ replications, each group of sets may be confounded once and once only in this manner, and in this case equal information will be obtained on every degree of freedom, and therefore every varietal comparison will be made with equal accuracy.

When p equals 5, for instance, the four squares given in Table I form a completely orthogonal set.

Table I. *Orthogonal set of* 5×5 *squares*

Square 1					Square 2					Square 3					Square 4				
a	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*	*a*	*b*	*c*	*d*	*e*
e	*a*	*b*	*c*	*d*	*d*	*e*	*a*	*b*	*c*	*c*	*d*	*e*	*a*	*b*	*b*	*c*	*d*	*e*	*a*
d	*e*	*a*	*b*	*c*	*b*	*c*	*d*	*e*	*a*	*e*	*a*	*b*	*c*	*d*	*c*	*d*	*e*	*a*	*b*
c	*d*	*e*	*a*	*b*	*e*	*a*	*b*	*c*	*d*	*b*	*c*	*d*	*e*	*a*	*d*	*e*	*a*	*b*	*c*
b	*c*	*d*	*e*	*a*	*c*	*d*	*e*	*a*	*b*	*d*	*e*	*a*	*b*	*c*	*e*	*a*	*b*	*c*	*d*

(The law of formation, which is the same for all prime numbers, should be obvious from inspection of this table. For $p=4$, 8 and 9, orthogonal sets are given in (1).)

If there are 25 varieties and these are numbered 1–25 *at random*, the first row of each square may be taken to represent the varieties 1–5, and so on. So long as every group of sets is confounded equally it is immaterial which are confounded in each replication. If we confound the groups corresponding to rows and columns in the first, those corresponding to squares

1 and 3 in the second and those corresponding to squares 2 and 4 in the third replication, then the first replication must be arranged on the ground in a square pattern so that the varieties 1–5 come in one row (not necessarily the first), the varieties 6–10 in another and so on. At the same time varieties 1, 6, 11, 16, 21 must come in one column, varieties 2, 7, 12, 17, 22 in another and so on. We must in fact randomize the rows and columns of the square:

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

This randomization process is that adopted in ordinary Latin squares in order to ensure an unbiased estimate of error.

In the second replication the varieties corresponding to the a's of the first square, namely varieties 1, 7, 13, 19, 25, must come in one row, and so on. At the same time the varieties corresponding to the a's of the third square, namely, 1, 9, 12, 20, 23, must come in one column, and so on. We must therefore randomize the rows and columns of the square:

		Square 3				
		a	*b*	*c*	*d*	*e*
Square 1	*a*	1	13	25	7	19
	b	20	2	14	21	8
	c	9	16	3	15	22
	d	23	10	17	4	11
	e	12	24	6	18	5

the structure of this square being given by the marginal letters.

Similarly the third replication is obtained by the randomization of the rows and columns of the square:

		Square 4				
		a	*b*	*c*	*d*	*e*
Square 2	*a*	1	15	24	8	17
	b	18	2	11	25	9
	c	10	19	3	12	21
	d	22	6	20	4	13
	e	14	23	7	16	5

The statistical analysis of balanced arrangements such as this is very simple. It is first necessary to calculate for each variety a quantity pQ, equal to p times the sum of the yields of all the plots of that variety, less the totals, $p+1$ in number, of every row and every column in which that variety occurs. The varietal differences, in terms of the yield of a single plot, are then given by the differences of the quantities

$$\frac{2}{p-1}Q \quad \text{or} \quad \frac{2}{p(p-1)}pQ.$$

A quantity equal to $\frac{2p}{p-1}$ times the mean yield should be added to each in order that their mean should equal the general mean.

The standard error of each *difference* is

$$\sqrt{2} \times \sqrt{\frac{2}{p-1}} \times \text{the standard error of a single plot.}$$

The sum of squares due to varieties in the analysis of variance is

$$\frac{2}{p-1}\,\mathrm{dev}^2\,Q \quad \text{or} \quad \frac{2}{p^2(p-1)}\,\mathrm{dev}^2\,pQ,$$

where $\mathrm{dev}^2\,Q$ indicates the sum of the squares of the deviations of Q from their own mean. The first and last of these formulae are given in two forms, the first form being most easily remembered, the second being that required for computation.

The remainder of the analysis of variance proceeds in the ordinary manner, items for rows, columns and squares being included to allow for the fertility differences eliminated by the design.

It should be noted that each of the above expressions can be derived from the parallel formula applicable to an experiment with the same number $\frac{1}{2}(p+1)$ of replications arranged in ordinary randomized blocks of p^2 plots, by writing $\frac{2}{p-1}$ for $\frac{2}{p+1}$ and replacing the sum of the yields of each variety by the corresponding Q.

It follows from this that the efficiency factor of the arrangement is

$$\frac{p-1}{p+1}.$$

This factor represents the loss of efficiency that would result if there were no gain in accuracy by the elimination of fertility differences between the rows or between the columns.

In this section we have only considered arrangements in which every set of degrees of freedom is confounded equally. Such arrangements may be called *balanced* quasi-Latin squares. This balance is analogous to the balance of designs in symmetrical incomplete randomized blocks (4). Sets of squares which lack this balance are also feasible, and are of interest in such cases as 8×8, which requires nine replications for complete balance, but in which nearly complete balance can be attained with four replications.

Such arrangements lose very little in efficiency through the slight lack of balance, the efficiency factors in the case of 8×8 squares being $\frac{7}{9}$ and $\frac{27}{35}$ or 0·778 and 0·771 respectively. The computations are somewhat more complicated, owing to the fact that the set of row and column totals entering into a single Q is no longer completely balanced for varieties, and an additional term must therefore be introduced to restore this balance.

III. Numerical example

We will take as an example the uniformity trial on oranges reported by Parker and Batchelor, the results of which have already been used to illustrate quasi-factorial arrangements in randomized blocks. The mean yields of the first six years for the whole trial are given in Table V of (3). The yields (less 100) of the first fifteen plots of each of the first five blocks are reproduced in Table II. The table also shows a superimposed arrangement of twenty-five varieties, indicated by the small numbers, which is the result of randomizing the rows and columns of the three squares given in the preceding section. The orientation of each square has also been allotted at random.

Table II. *Yields (less 100) and arrangement of varieties*

Plot	Block *M*	*L*	*K*	*J*	*I*	Total
2	-13^{23}	3^{21}	-4^{25}	3^{22}	13^{24}	2
4	-5^{8}	6^{6}	19^{10}	4^{7}	20^{9}	44
6	11^{18}	10^{16}	9^{20}	-1^{17}	40^{19}	69
8	-21^{3}	12^{1}	-1^{5}	15^{2}	29^{4}	34
10	-2^{13}	13^{11}	16^{15}	10^{12}	22^{14}	59
Total	-30	44	39	31	124	208
12	2^{25}	9^{6}	22^{14}	28^{17}	13^{3}	74
14	-7^{7}	26^{18}	37^{21}	28^{4}	8^{15}	92
16	0^{19}	16^{5}	29^{8}	19^{11}	13^{22}	77
18	-7^{13}	15^{24}	12^{2}	19^{10}	20^{16}	59
20	-2^{1}	16^{12}	27^{20}	35^{23}	25^{9}	101
Total	-14	82	127	129	79	403
22	-2^{10}	38^{19}	32^{3}	37^{21}	38^{12}	143
24	3^{22}	$(18)^{6}$	21^{20}	19^{13}	15^{4}	76
26	-3^{18}	21^{2}	4^{11}	10^{9}	17^{25}	49
28	19^{1}	19^{15}	29^{24}	32^{17}	29^{8}	128
30	8^{14}	31^{23}	48^{7}	30^{5}	52^{16}	169
Total	25	127	134	128	151	565

The yield of the plot 24 of the block L was missing from the original records. The omission of this row entirely would be unduly unfavourable to the Latin square design, since the whole row is low-yielding, and the row has therefore been retained, with the value 118 for the missing yield, calculated from the row and column values of the third square.

The quantities $5Q$ are shown in Table III (negative signs being omitted). Thus, for example,

$$5Q_1 = 5\times 12 - 44 - 34 + 5\times(-2) + 14 - 101 + 5\times 19 - 25 - 128 = -173.$$

Table III

Varieties	Values of $-5Q$				
1–5	173	187	314	246	304
6–10	282	231	232	250	259
11–15	312	247	228	318	309
16–20	161	264	117	136	261
21–25	151	195	233	244	226

The adjusted yields of the varieties are shown in Table IV. Thus, for example, the adjusted yield of variety 1 equals

$$\frac{10}{4}(\text{mean yield}) + \frac{2}{5.4}\,5Q_1 = 139{\cdot}2 - \frac{1}{10}(173) = 121{\cdot}9.$$

Table IV

Varieties	Adjusted varietal means				
1–5	121·9	120·5	107·8	114·6	108·8
6–10	111·0	116·1	116·0	114·2	113·3
11–15	108·0	114·5	116·4	107·4	108·3
16–20	123·1	112·8	127·5	125·6	113·1
21–25	124·1	119·7	115·9	114·8	116·6

The analysis of variance is shown in Table V. The sum of squares for varieties is given by the sum of the squares of the deviations of the values of Table III divided by $25.4/2$ or 50.

Table V. *Analysis of variance*

	D.F.	Sums of squares	Mean square
Squares	2	2556·24	1278·12
Rows	12	2696·08	224·67
Columns	12	7108·08	592·34
Varieties	24	1566·64	65·28
Error	24	1381·28	57·55
Total	74	15308·32	

The mean square for varieties is 65·28, and that for error is 57·55. Nothing has been deducted for the missing plot, since the yield of this was determined independently of the varietal arrangement. The mean squares for varieties and error are approximately equal, as they should be.

The standard error assignable to each of the adjusted values of Table IV is given by

$$\sqrt{(\tfrac{2}{4} \times 57{\cdot}55)} = 5{\cdot}36.$$

IV. Relative efficiency of various arrangements

The higher efficiency of Latin squares compared with randomized blocks is likely in general to more than compensate for the lower efficiency factors of quasi-Latin squares. It will be recalled that the efficiency factors for two-dimensional quasi-factorial arrangements (p^2 varieties) are $\dfrac{p+1}{p+3}, \dfrac{p+1}{p+2\frac{1}{2}}$ and $\dfrac{p}{p+1}$ according as two, three or $p+1$ groupings are used. The numerical values of these factors, and of the factor $\dfrac{p-1}{p+1}$ for quasi-Latin squares, are shown in Table VI.

Table VI. *Efficiency factors*

No. of varieties	16	25	49	64	81	121	169
p	4	5	7	8	9	11	13
Minimum no. of replications for quasi-Latin squares	5	3	4	9	5	6	7
Quasi-Latin squares	0·6	0·667	0·75	0·778	0·8	0·833	0·857
Quasi-factorials in blocks:							
Two groupings	0·714	0·75	0·8	0·818	0·833	0·857	0·875
Three groupings	0·769	0·8	0·842	0·857	0·870	0·889	0·903
$p+1$ groupings	0·8	0·833	0·875	0·889	0·9	0·917	0·929

All values of p from $p=4$ to $p=13$ have been included for which a completely orthogonal set of squares is known to exist. The even values $p=4$ and $p=8$ will require $p+1$, i.e. five and nine replications in order to obtain a balanced arrangement. In many cases, of course, some multiple of the minimum number of replications will be required to attain the desired accuracy.

It will be noted that for quasi-factorial arrangements in blocks the balanced arrangement with $p+1$ groupings requires $p+1$ replications. It is an additional advantage of quasi-Latin squares that when p is odd balance is attained with half the number of replications required for quasi-factorial arrangements in randomized blocks. The attainment of balance has three advantages. The efficiency is maximized, the standard errors of all varietal comparisons are the same, and the computations are simplified.

The efficiency of the various arrangements in the example of the preceding section may now be considered. Table VII gives the residual mean squares after eliminating squares only, squares and columns or rows, and squares, columns and rows. These are the mean error mean squares that will be obtained in arrangements in randomized blocks of 25, in arrangements in randomized blocks of 5, and in 5×5 Latin squares respectively.

Table VII. *Residual mean squares*

	D.F.	Mean square	Information
Blocks of 25	71	179·61	1
Blocks of 5 (columns)	59	95·66	1·88
Blocks of 5 (rows)	59	170·44	1·05
5×5 Latin squares	47	62·72	2·86

In each case one degree of freedom has been deducted to allow for the missing plot. This procedure is approximate except in the case of the Latin squares.

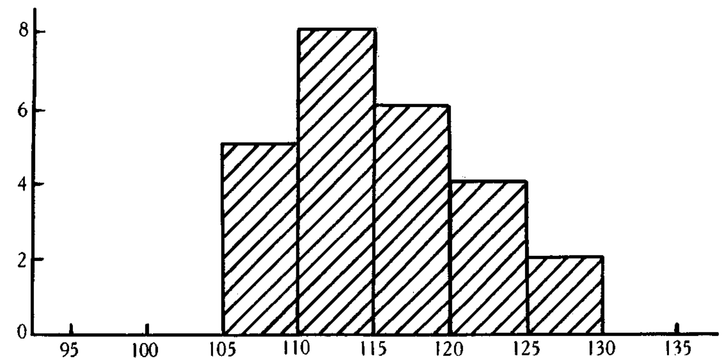
This table provides an excellent illustration of the power of the Latin square design in eliminating fertility differences. Although in this case the elimination of row effects alone would scarcely have reduced the residual variance, their elimination subsequent to that of columns has effected a substantial reduction. In general it is easy to see that, if the variance due to rows is equal to that due to columns, the elimination of both rows and columns will bring about a relative reduction of the residual variance of more than twice that due to the elimination of either alone.

Multiplying the relative amounts of information per plot of Table VII by the efficiency factors of Table VI, we obtain the relative efficiencies shown in Table VIII. Thus in this particular example the use of quasi-Latin squares instead of randomized blocks containing all the varieties almost doubles the information obtained.

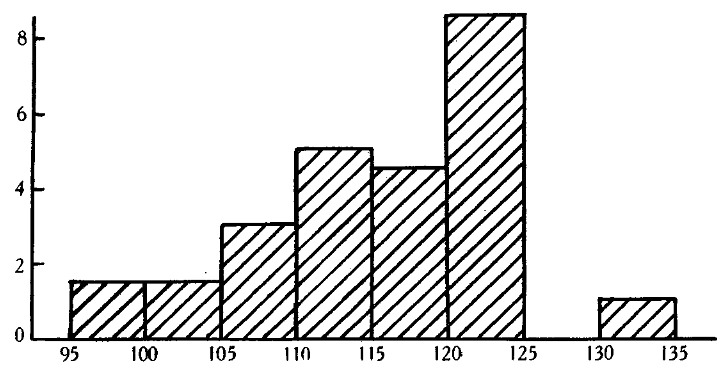
In order to illustrate the increase in precision resulting from the use of quasi-Latin squares, an arrangement of twenty-five varieties in randomized blocks of twenty-five plots (corresponding to the squares already used) was also superimposed on the variety trial. The varietal means so obtained are shown in Table IX. Their variability is easily seen to be considerably greater than that of the adjusted varietal means of Table IV, the ranges in the two cases being from 99·7 to 131·3, and from 107·4 to 127·5. The two distributions of values

Table VIII. *Relative efficiencies of various arrangements*

	Efficiency in chosen example	Efficiency when there are no fertility differences to eliminate
Randomized blocks of 25 plots	100	100
Quasi-factorial arrangements in blocks of 5 plots:		
Two groupings	140·8	75
Three groupings	150·2	80
Six groupings	156·5	83·3
Quasi-Latin squares	190·9	66·7



(*a*) Arrangement in quasi-Latin squares (Table IV).



(*b*) Arrangement in randomized blocks of twenty-five plots (Table IX).

Fig. 1. Distributions of the twenty-five varietal means from Table IV and Table IX.

are also compared graphically in Fig. 1. (Both sets of values happen to give a fair representation of the amount of variation that would be obtained on the average in this trial from the two types of arrangement.)

Table IX. *Varietal means from arrangements in randomized blocks of twenty-five plots*

104·3	118·7	120·0	112·0	111·7
114·0	122·7	109·3	124·3	108·7
107·7	122·3	119·7	99·7	115·3
100·0	121·3	121·0	122·0	112·3
131·3	117·3	120·7	121·0	114·7

In this example quasi-Latin squares have also proved markedly more efficient than quasi-factorial arrangements in randomized blocks, though the example cannot be regarded as particularly favourable to the Latin square arrangement, since blocks account for the greater part of the fertility irregularities. In general it may be doubted whether quasi-factorial arrangements in randomized blocks are likely to result in any great gain in efficiency when the number of varieties is as small as twenty-five. It would appear, however, that even with this small number of varieties quasi-Latin squares are likely to be very effective. It has been found, for example, that in the Rothamsted experiments and experiments at associated centres from 1927 to 1934 the error variance of 5×5 Latin squares was reduced on the average in the ratio of 2·49 : 1 from what it would have been if the experiments had been completely randomized. This, multiplied by the efficiency factor $\frac{2}{3}$, gives an average increase of 66 per cent in the information when 5×5 quasi-Latin squares instead of randomized blocks of twenty-five plots are used for varietal trials involving twenty-five varieties.

Latin squares are, of course, only suited to certain types of variety trial. With crops that are sown by drill the practical requirements of drilling may necessitate long narrow plots, and preclude the use of a Latin square design. In such crops as fruit, however, this consideration does not hold, and even with long narrow plots the additional restrictions of a Latin square are often strikingly effective in reducing the error variance.

V. The use of quasi-Latin squares in three-dimensional quasi-factorial designs

If a number p^3 of varieties is appropriately divided into three groups of p^2 sets of p varieties each, and each of these sets is arranged in one or more randomized blocks, a three-dimensional quasi-factorial arrangement results. The division into the three groups of sets may be effected by setting out the varieties at random in a cube and taking the sets lying on lines parallel to the edges of the cube. The analysis of such arrangements was discussed in (3), where it was shown that the efficiency factor was $\dfrac{2(p^2+p+1)}{2p^2+5p+11}$.

In certain cases quasi-Latin squares can be used as the basis of an arrangement of this type, for if the varieties be divided into p sets of p^2 varieties in two ways, orthogonal to one another, the members of each set can be compared by means of a set of quasi-Latin squares. The appropriate division can be effected by taking the sets lying on planes parallel to two of the faces of a random cube of the varieties. If sufficient replications are available, the group of sets corresponding to planes parallel to the third face may also be taken, but there is little further gain in efficiency. The efficiency factors for the arrangements using two and three groupings are

$$\frac{p-1}{p+1}\cdot\frac{p^2+p+1}{p^2+p+3} \quad \text{and} \quad \frac{p-1}{p+1}\cdot\frac{p^2+p+1}{p^2+p+2\frac{1}{2}}$$

respectively.

Arrangements of this type compare favourably with ordinary three-dimensional quasi-factorial arrangements in randomized blocks, for the advantages of Latin square design are obtained without any great reduction in the efficiency factor below that of the randomized block arrangements. Table X gives the numerical values of the efficiency factors for the various types of arrangement for $p=4$, 5 and 7.

Table X. *Efficiency factors for three-dimensional arrangements*

No. of varieties	64	125	343
p	4	5	7
Minimum no. of replications, two groupings	10	6	8
Quasi-Latin squares:			
Two groupings	0·548	0·626	0·725
Three groupings	0·560	0·636	0·731
Randomized blocks	0·667	0·721	0·792

Whether these arrangements are likely to be more efficient than the equivalent two-dimensional arrangements in quasi-Latin squares depends on the additional reduction in variance that results from the reduction in size of the Latin squares.

The estimation of the varietal differences is best carried out in two stages. The quantities Q are first calculated for the various sets of quasi-Latin squares, and from them estimates of the varietal means are obtained for each grouping. These estimates can then be set out in three-way tables (one for each grouping) and an adjusted table prepared in a similar manner to that adopted in quasi-factorial designs in randomized blocks. With a similar factorial notation to that previously used in (3) the adjusted varietal means t_{uvw} are given by

$$t_{uvw}=\tfrac{1}{2}(x_{uvw}+y_{uvw}-\bar{x}_{u..}+\bar{x}_{.v.}+\bar{y}_{u..}-\bar{y}_{.v.}),$$

when the two groups of sets are formed by holding u and v constant respectively. This formula is almost the same as that for two-dimensional quasi-factorial arrangements in randomized blocks on p. 433 of (3). In the case of three groupings the formula for t_{uvw} is identical with that given on p. 438 for two-dimensional quasi-factorial arrangements in three groups of sets.

The comparisons between the t_{uvw} are not all of exactly the same precision, the variances in the case of two groupings of the differences of two t's of varieties occurring together in two, one or no sets of quasi-Latin squares being

$$\frac{p+1}{p-1}, \quad \frac{p+1}{p-1}\cdot\frac{p^2+1}{p^2}, \quad \frac{p+1}{p-1}\cdot\frac{p^2+2}{p^2}$$

respectively times the corresponding variance when there are no restrictions and the error mean square is unchanged. Thus even with two groupings the variation in precision is so small that the mean error variance given by the efficiency factor will suffice for all practical purposes (except possibly for varieties having both sets of quasi-Latin squares in common). In the case of three groupings the variation in precision will be even smaller.

The analysis of variance involves no new principle, the procedure to be followed being a combination of the ordinary procedure for quasi-Latin squares, and the procedure for quasi-factorial arrangements in randomized blocks.

VI. The use of the quasi-Latin squares principle in factorial design

In varietal trials we are equally interested in comparisons between every pair of varieties, and consequently the aim of the design is to confound all comparisons equally frequently. In experiments involving several factors, however, we are usually less concerned with the high order interactions than with the effects of single factors and interactions between two factors only. We may consequently be prepared to sacrifice some or all of the information on one or more of the high order interactions, provided that the efficiency of the remaining comparisons is thereby increased.

If the quasi-Latin square type of design is used for an experiment involving several factors, therefore, the condition that every set of degrees of freedom is confounded equally may be dispensed with. Instead we may confine the confounding to sets of degrees of freedom representing high order interactions, keeping the main effects free from confounding.

If, for example, instead of sixty-four varieties we have an experiment including all combinations of eight varieties and two levels of each of the three standard fertilizers, i.e. an $8 \times 2 \times 2 \times 2$ factorial design, the treatment degrees of freedom will partition into

Varieties	7
Fertilizers	7
Fertilizers × varieties	49

If we take the rows and columns of a completely orthogonal set of 8×8 squares to represent varieties and fertilizers respectively, seven groups of sets of the varietal and treatment combinations will be determined by these seven squares. Contrasts between sets of the same group will correspond to seven of the 49 degrees of freedom for interaction, the whole seven sets accounting for the whole 49 degrees of freedom. Each set of seven degrees of freedom will be found to be of the form

$$\begin{array}{lll} V_1.N & V_4.N.P & \\ V_2.P & V_5.N.K & V_7.N.P.K \\ V_3.K & V_6.P.K & \end{array}$$

where $V_1, V_2, \ldots, V_7$ are seven orthogonal degrees of freedom for varieties of the form

$$V_1 = v_1 + v_2 + v_3 + v_4 - v_5 - v_6 - v_7 - v_8,$$

etc., V_4 being given by the "interaction" of V_1 and V_2, etc.

In each replication two such sets may be confounded, one with the rows and one with the columns, so that if there are three replications $\frac{2}{3}$ of the relative information will be obtained on 42 of the 49 interaction degrees of freedom, and full information on the remainder.

A variant of this design is that in which there are only four varieties, so that we have a $4 \times 2 \times 2 \times 2$ factorial design. A single square will then give two complete replications, and one set of three degrees of freedom can be confounded with rows and another with columns. Sets of the type

Rows	Columns
$V_1.N.P$	$V_1.N.K$
$V_2.N.K$	$V_2.P.K$
$V_3.P.K$	$V_3.N.P$

are possible. In this case a single replication will sacrifice all the information on these degrees of freedom, the analysis being of the form:

Rows	7
Columns	7
Varieties	3
Fertilizers	7
Unconfounded interactions	15
Error	24
Total	63

This design is derivable from the $8 \times 2 \times 2 \times 2$ design by using duplicate varieties.

A simpler example of the same type of design is provided by the arrangement of a $2 \times 2 \times 2$ design in two or more 4×4 quasi-Latin squares, confounding two interaction degrees of freedom, one with rows and one with columns, in each square. The following two squares will confound the interactions shown:

(1)	np	nk	pk
npk	k	p	n
n	p	k	npk
pk	nk	np	(1)

Rows: $N.P.K$
Columns: $P.K$

(1)	k	np	npk
nk	pk	n	p
p	n	pk	nk
npk	np	k	(1)

Rows: $N.P$
Columns: $N.K$

Thus with two replications we might sacrifice half the information on each of the interaction degrees of freedom. Alternatively all information on $N.P.K$ and $P.K$ might be sacrificed, so as to obtain full precision on $N.P$ and $N.K$.

Similar designs are possible with factors at three levels. In a $3 \times 3 \times 3$ experiment with three replications, for instance, we may confound one pair of degrees of freedom for the interaction between the three factors with the rows, and a second pair with the columns.

Partial confounding within the limits of a single square is also possible, provided of course that the square comprises more than a single replication. Thus in a 2^5 design in a single 8×8 square eight of the ten three-factor interactions, and four of the five four-factor interactions, may be partially confounded, one half the relative information being obtained on each.

In all such designs it can easily be shown that the randomization of rows and of columns provides an unbiased estimate of error. Since, however, the original squares are of a rather special type, the designs even after randomization possess certain systematic elements which are at first sight disconcerting. Thus in the $3 \times 3 \times 3$ design the three levels of any one factor (or numbers representing the contrasts corresponding to the unconfounded interactions) can be brought into the pattern given by Fig. 2 (*a*), and a typical pattern after randomization is that of Fig. 2 (*b*):

(*a*)

0	0	0	1	1	1	2	2	2
0	0	0	1	1	1	2	2	2
0	0	0	1	1	1	2	2	2
1	1	1	2	2	2	0	0	0
1	1	1	2	2	2	0	0	0
1	1	1	2	2	2	0	0	0
2	2	2	0	0	0	1	1	1
2	2	2	0	0	0	1	1	1
2	2	2	0	0	0	1	1	1

(*b*)

2	1	0	0	0	1	2	1	2
1	0	2	2	2	0	1	0	1
0	2	1	1	1	2	0	2	0
0	2	1	1	1	2	0	2	0
1	0	2	2	2	0	1	0	1
0	2	1	1	1	2	0	2	0
2	1	0	0	0	1	2	1	2
2	1	0	0	0	1	2	1	2
1	0	2	2	2	0	1	0	1

Fig. 2

It is not, perhaps, very likely that the z distribution will be appreciably disturbed, but it would be satisfactory to have confirmation of this point by developing the distribution by randomization over a series of uniformity trials.

VII. Summary

The principles of quasi-factorial design are extended so as to enable trials involving a number of varieties or treatments which is a perfect square (not 6^2 or some other numbers, however) to be so arranged that differences associated with two groupings of the experimental material, such as the rows and columns of an agricultural field trial, are simultaneously eliminated from the varietal comparisons.

As a numerical example a quasi-Latin square design for 25 varieties is superimposed on the uniformity trial on oranges which was used in a previous paper to illustrate quasi-factorial designs in randomized blocks. A gain in efficiency over an arrangement in ordinary randomized blocks of 91 per cent resulted, the corresponding gain in a quasi-factorial design in randomized blocks (two groupings) being 41 per cent.

Various other possible applications of the quasi-Latin square principle are briefly discussed.

Note on Nomenclature

Since the above paper was written, I have provided an alternative and shorter nomenclature for quasi- (or pseudo-) factorial designs, using the term *lattice*, which enables the various types of design to be described very concisely. The following table of equivalents will make clear the sense in which the word is used.

Two-dimensional quasi-factorial designs in randomized blocks:	
In two equal groups of sets	Lattice *or* square lattice
In two unequal groups of sets	$p \times q$ lattice
In three (equal) groups of sets forming a Latin square	Triple lattice
In $p+1$ (equal) groups of sets	Balanced lattice
Three-dimensional quasi-factorial designs in randomized blocks:	
In three equal groups of sets	Cubic lattice or three-dimensional lattice
In three unequal groups of sets	$p \times q \times r$ lattice
Balanced set of quasi-Latin squares	Lattice squares
Three-dimensional quasi-factorial design in quasi-Latin squares	Cubic lattice in lattice squares

The terms *quasi-factorial* and *quasi-Latin square* may be usefully retained as general descriptive terms. In particular the term *quasi-Latin square* appears specially appropriate for the factorial designs outlined in section VI. Various designs of this latter type have been developed in detail in (5).

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