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# Distance metric choice can both reduce and induce collinearity in geographically weighted regression

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## Abstract

This paper explores the impact of different distance metrics on collinearity in local regression models such as Geographically Weighted Regression (GWR). Using a case study of house price data collected in Hà Nội, Vietnam, and by fully varying both power and rotation parameters to create different Minkowski distances, the analysis shows that local collinearity can be both negatively and positively affected by distance metric choice. The Minkowski distance that maximised collinearity in GWR was approximate to a Manhattan distance with (power = *0.70*) with a rotation of *30*°, and that which minimised collinearity was parameterised with power = *0.05* and a rotation of *70*°. The results indicate that distance metric choice can provide a useful extra tuning component to address local collinearity issues in spatially varying coefficient modelling and that understanding the interaction of distance metric and collinearity can provide insight into the nature and structure of the data relationships. The discussion considers first, the exploration and selection of different distance metrics to minimise collinearity as an alternative to localised ridge regression, lasso and elastic net approaches. Second, it discusses the how distance metric choice could extend the methods that additionally optimise local model fit (lasso and elastic net) by selecting a distance metric that further helped minimise local collinearity. Third, it identifies the need to investigate the relationship between kernel bandwidth, distance metrics and collinearity as an area of further work.

**Key words**: GWR; distance metrics; model fit; collinearity.

## 1. Introduction

Geographically weighted regression (GWR) is a technique used to explore spatially-varying data relationships (Brunsdon et al. 1996, Fotheringham et al. 2002). Its original conception reflected a desire to move beyond from global, 'whole map' (Openshaw 1996) and 'one-size-fits-all' (Fotheringham and Brunsdon 1999) statistics to ones that captured and reflected local process heterogeneity. This was reflected in Goodchild's (2004) proposal for a second law of geography, the principle of spatial heterogeneity or nonstationarity, in which he noted the lack of a 'concept of an average place on the Earth's surface comparable, for example, to the concept of an average human' (Goodchild 2004, p302). For regression, and from a nonstationary relationship viewpoint, a number of localised approaches have been developed including the expansion method (Casetti 1972), weighted spatial adaptive filtering model (Gorr and Olligschlaeger 1994), GWR, Bayesian space-varying coefficient (SVC) models (Gelfand et al. 2003; Assunção 2003) and re-focused versions of eigenvector spatial filtering (ESF) (Griffith 2008; Murakiami et al. 2017). Other aspects of nonstationarity in regression can also be considered, such as those centred around the error term (e.g. Paez et al. 2002a, 2002b; Harris et al. 2010; 2011a). In particular, GWR has been the most widely applied localised regression in geographical analyses. It has conceptual simplicity: as geographers, we implicitly expect processes and relationships to vary locally and not to be the same everywhere. Rather, we acknowledge that the relationship among predictor and response variables may change over space. GWR provides a tool to identify and explore these varying relationships. The originators have long supported different implementations, either as standalone (e.g. GWR3.x Charlton et al. 2003) or as packages (e.g. R package GWmodel by Lu et al. 2014a, Gollini et al. 2015). It has been also incorporated as a tool in the most popular GIS software (ESRI 2009).

The operation of GWR, and other geographically weighted (GW) models such as GW principle components analysis (Harris et al. 2011b), involves performing location-wise calibrations using subsets of the data (observations) around each location. Observations nearest to the calibration point are given the greatest weight, while data points beyond a certain threshold distance (i.e. bandwidth) are given a negligible or zero weight, depending on the distance-decay function used. GWR has certain elegance. First, it reflects public and scientific intuition about and experience of spatial variation: birds of a feather *do* flock together and most anthropogenic and environmental processes cluster rather than being evenly or randomly distributed. Second, the weighting scheme describes a distance-decay commonly found by observation and measurement of geographical processes, and implicitly reflected in Tobler's first law of geography (Tobler 1970).

There are a number of critical considerations in any GW analysis, the most important of which is bandwidth selection as this determines how many data points could be included in each local calculation and the associated degree of smoothing in the model outputs. So for a GWR model, the bandwidth determines the degree of variation in local regression coefficient estimates. Bandwidths can be a fixed distance or an adaptive distance, where that latter defines a fixed number of nearest data points. Bandwidth calibration routines exist to determine the optimal (fixed or adaptive) bandwidth by maximising some measure of model fit such as Akaike information criterion (Akaike 1973) or leave-one-out cross validation (e.g. Cleveland 1979; Bowman 1984; Brunsdon et al. 1996). Further studies for selecting bandwidths in GWR include Páez (2004) for anisotropic bandwidths, Farber and Páez (2007) for a robust bandwidth selection, Brunsdon et al. (1999) and Nakaya et al. (2005) for bandwdiths in mixed GWR, Fotheringham et al. (2017) and Lu et al. (2017a) for scale-dependent bandwidths and Fotheringham et al. (2015) for spatio-temporal bandwidths. A second important concern is the nature of the weighting scheme, which is implicitly connected with the selection of a distance metric. Choice of distance metric in GWR has been advanced by Lu et al. (2014a, 2015, 2016, 2017a) who used network distance, travel times, Minkowski distances and parameter-specific distance metrics to improve GWR model fit.

GWR has been criticised for its poor inferential properties, which primarily stems from GWR being a collection of local models, where data are partially re-used from neighbouring local models, and where no single non-stationary model exists, unlike, say, a Bayesian SVC model which itself, is based on a multivariate geostatistical construct (e.g. see Finlay 2011). Similarly, Griffith (2008) notes the degrees-of-freedom problem, describing GWR as a 'brute-force version of indirect spatial filtering'. Despite these valid criticisms, a considered application of GWR can still provide an important and robust exploratory tool, and benefits from an inherent simplicity.

A criticism of GWR however, that has attracted much attention relates to that of collinearity, as first articulated by Wheeler and Tiefelsdorf (2005), and first addressed by Wheeler (2007). Collinearity occurs when pairs of predictor variables have a strong positive or negative relationship between each other. Strong collinearity can affect model reliability and precision and can result in unstable coefficient estimates, inflated standard errors and inferential biases (Dormann et al. 2013). As a result, model extrapolation may be erroneous and there may be problems in separating variable effects (Meloun et al. 2002). Various approaches exist to address collinearity in regression modelling, such as partial least squares regression, principal component analysis regression, ridge regression (Hoerl 1962, Hoerl and Kennard 1970) and extensions, such as the lasso (Tibshirani 1996) and the elastic net also provide predictor variable sub-set selection (Zou and Hastie 2005).

In GWR, collinearity may be observed in the local subsets of the data under the kernel even when they are not observed globally (Wheeler and Tiefelsdorf 2005, Wheeler 2007). Accounting for the presence of local collinearity is highly recommended in any GWR analysis and options include a globally-defined ridge GWR (Wheeler 2007), a locally-defined ridge GWR (Brunsdon et al. 2012), GWR models where the kernel bandwidth is locally increased in areas of strong collinearity (Brunsdon et al. 2012; Bárcena et al. 2014) and a GW lasso (Wheeler 2009; Yoneoka et al. 2016). Wheeler (2007), Lu et al. (2014b) and Gollini et al. (2015) describe sets of localised diagnostic tools that can gauge the nature of collinear effects in any given GWR calibration, such as the mapping of local matrix condition numbers. Such diagnostics are used in this study.

In summary, a GWR undertakes a series of local regressions. Subsets of data are borrowed from nearby locations and their contribution is weighted by their distance to the location under consideration. The kernel bandwidth determines how much data is included in each subset for each local regression. The distance metric in GWR is commonly Euclidean, but any metric is possible and a carefully chosen distance metric and associated bandwidth can improve model fit (e.g. Lu et al. 2014a). Collinearity affects model reliability and precision and may be present locally in GWR even when not observed in the global regression. Although adaptations of GWR are available to cater for collinearity (e.g. ridge, lasso), as yet no research has considered the impacts of the choice of distance metric on collinearity. Whilst Lu et al. (2014a, 2015, 2016, 2017a) have considered different distance metrics these have all been considered with respect to model fit. This paper explores the impacts of different Minkowski distances, where the power and rotation parameters are allowed to vary, and road network distance on local measures of collinearity using a house price case study in Hà Nội, Vietnam.

## 2. Methods

## 2.1 Data

A detailed house price survey of nearly 1,000 households sampled from a 400m x 400m grid was undertaken in 2014 in Hà Nội, Vietnam. It collected data for a range of predictor variables, describing property characteristics including:

- Number of residents (TotNum)
- Years in education of householder (EdYears)
- Number of separate bedrooms (SepBed)
- Ground floor area ($m^2$) (GFA)
- Plot area ($m^2$) (PlotArea)
- Length of frontage (m) (Frontage)
- Perceived travel time to city centre (minutes) (TimeCity)
- House price per square metre (VND $m^{-2}$) (Ppsqm)
- Euclidean distance to city centre (m) (DistCity)

The data were cleaned for missing and NULL variables and the result was 558 data points. The nine predictor variables were used to model house price in millions of Vietnamese Dong. The spatial distribution of the data points is shown in Figure 1.

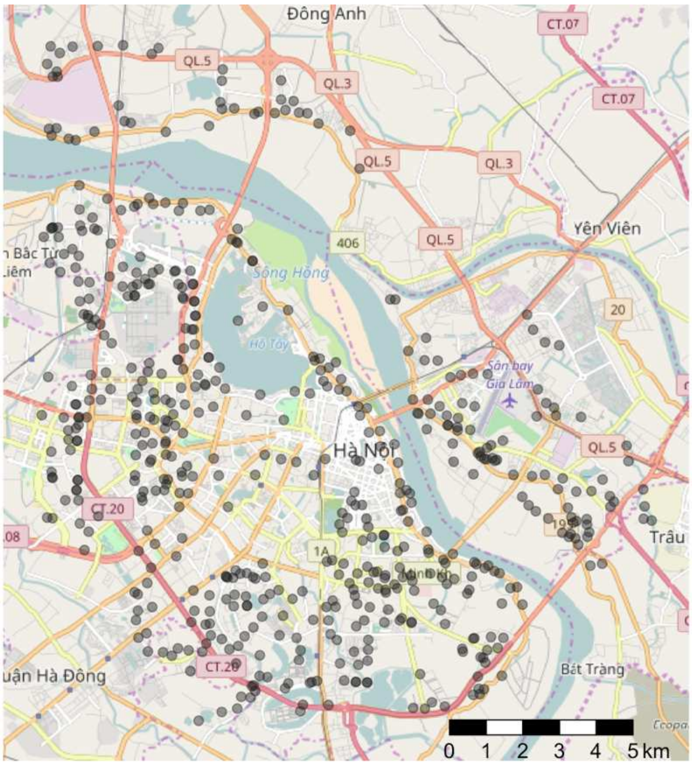


Figure 1. The study area in Hà Nội, the 558 survey data points with a small transparency to show their density and an OpenStreetMap backdrop.

### 2.2 Analysis

The study aim was to explore the impacts of different distance metrics in a GWR model with respect to collinearity. The data points in Figure 1 were used as the GWR calibration points. GWR is similar to ordinary least squares (OLS) regression, but is an extension to the spatial domain. As OLS is a non-spatial model, it can be defined as:

$$y_i = \beta_0 + \sum_{k=1}^{m} \beta_k x_{ik} + e_i \tag{1}$$

where for observations indexed by $i = 1, \ldots, n$, $y_i$ is the response variable (house price), $x_{ik}$ is the value of the $k^{th}$ predictor variable (house characteristic), $m$ is the number of predictor variables, $\beta_0$ is the intercept term, $\beta_k$ is the regression coefficient for the $k^{th}$ predictor variable and $e_i$ is the random error term.

**Geographically weighted regression**

The basic form of GWR is similar to that given in Equation 1, but with locations associated with the model coefficient terms:

$$y_i = \beta_0(u_i, v_i) + \sum_{k=1}^{m} \beta_k(u_i, v_i) x_{ik} + e_i \tag{2}$$

where $(u_i, v_i)$ is the spatial location of the $i^{th}$ observation and $\beta_k(u_i, v_i)$ is a realization of the continuous function $\beta_k(u, v)$ at point $i$. In matrix terms, the coefficients of GWR are estimated from:

$$\hat{\boldsymbol{\beta}}(u_i, v_i) = \left(\mathbf{X}^{\mathrm{T}}\mathbf{W}(u_i, v_i)\mathbf{X}\right)^{-1} \mathbf{X}^{\mathrm{T}}\mathbf{W}(u_i, v_i)\mathbf{y} \tag{3}$$

where $\mathbf{W}(u_i, v_i)$ is a $(n \times n)$ spatial weighting diagonal matrix determined from the kernel function specified below, $\mathbf{X}$ is a $(n \times (m+1))$ predictor data matrix and $\mathbf{y}$ is a $(n \times 1)$ response data vector.

### Kernel bandwidth selection

For this study, an optimum bandwidth for GWR was found by minimizing a model fit diagnostic via a leave-one-out cross-validation (CV) score (Bowman 1984, Brunsdon et al. 1996). Furthermore, a bi-square weighting kernel was always specified, where for each data point under this kernel (of a given bandwidth), a weight $w_{i,j}$ was calculated based on its distance to the kernel centre as follows:

$$w_{i,j} = \begin{cases} 1 - \left(\frac{(d_{i,j})^2}{b^2}\right), d_{i,j} < b \\ \quad 0, \qquad otherwise \end{cases} \tag{4}$$

where $d_{i,j}$ is the distance in from the kernel centre to the data point $P_j$ and $b$ is the bandwidth.

### Collinearity

The preceding section describes the generic construction of a GWR model. In order to understand the impacts of distance metric choices on collinearity, different distance metrics were used to parameterise a sequence of GWR models, which were then subject to a collinearity diagnostic procedure that determined their local design matrix condition number (CN). Other measures of local collinearity are available, such as local predictor variable correlations, local variance decomposition proportions (VDPs) and local variance inflation factors (VIFs). A thorough investigation of local collinearity in a GWR analysis would report, local CNs, local correlations, local VDPs and local VIFs, as no individual diagnostic provides a full picture of collinearity (Wheeler 2007). For example, VIFs do not detect collinearity with the intercept (Wheeler 2010). However, local CNs have been found to provide a superior diagnostic for investigating local collinearity (Wheeler 2007) and only this diagnostic is reported here. Specifically, the CN of the local design matrix can be calculated using the method described in Belsley et al. (1980), and CNs greater than 30 are suggestive of likely collinearity issues amongst any one pair of predictor variables (heuristics from Belsley et al., 1980). Local CNs were generated using the GWR diagnostics procedure described in Gollini et al. (2015) and implemented using the *gwr.collin.diagno* function in the **GWmodel** R package (Lu et al. 2014b, 2017b, Gollini et al. 2015). This returns a local CN for each GWR calibration point.

**Distance metrics**

GWR and any GW model require some measure of distance in order to determine the weightings applied to the local data subsets. Typically, the default distance metric is Euclidian distance. Lu et al. (2016) describe the application of different Minkowski distances in GWR. Minkowski distance is a metric in vector space which can be considered as a generalization of Euclidean distance. In 2-dimensional Euclidian space, a generalized Minkowsi distance can be defined as:

$$d_{p\theta} = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}(|\sin(\theta + a)|^p + |\cos(\theta + a)|^p)^{\frac{1}{p}} \qquad (5)$$

where $(u_i, v_i)_{i=1,2}$ are the coordinates in Euclidean space, and the angle *a* is the rotation angle equalling to $tan^{-1}(\frac{(u_1-u_2)}{(v_1-v_2)})$.

Different Minkowski distances can be generated by varying the exponent or power parameter, *p*, together with the coordinate rotation angle, *θ*. Lu et al. (2016) note the difficulty in conceptualising any specific Minkowski distance, except for the common cases of Manhattan (*p = 1*), Euclidean (*p = 2*) and Chebyshev ($p = \infty$) distances, and that the rotation parameter *θ* adds to this difficulty. In this study, road network distances for Hà Nội and a sequence of Minkowski distances were investigated, with values of $p$ changing, but initially, the rotation angle *θ* was set to 0° in each case.

The value of $p$ can be any non-negative real number and *θ* can lie between 0 and π/2 in radians (i.e. between 0° and 90° rotation). Figure 2 compares Minkowski distances generated using different values of $p$ and *θ = 0*° with road network distance. The data are ordered on the *x*-axis by the smallest network distance and the plots show how different values of $p$ result in different ranges and distributions when compared to network distances. The closest to the Hà Nội road network distance is when $p$ *= 1* (Manhattan).

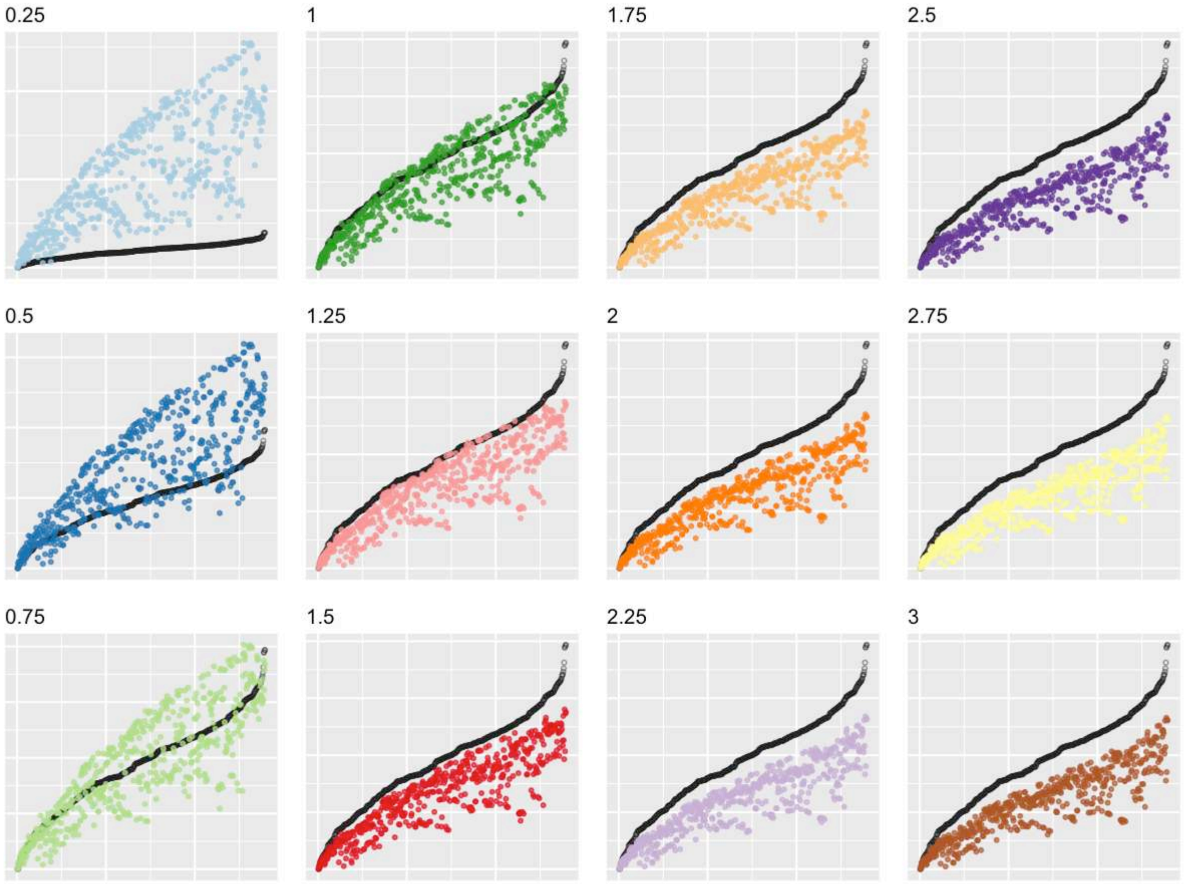


Figure 2. A comparison of different Minkowski distances using different power values with Hà Nội road network distances for the 558 data points, with rotation set to 0 and ordered on the x-axis by network distance.

## Implementation and code

An initial analysis was undertaken to investigate the impact of different values of $p$ on collinearity. A sequence of values for $p$ were generated ranging from 0.05 to 4 in steps of 0.05. The stages of analysis for each value of $p$ were as follows:

i. Calculate the Minkowski distances between each of the calibration points (558 data locations);

ii. Determine the optimal adaptive GWR bandwidth using the leave-one-out CV score approach using the Minkowski distances;

iii. Run a GWR diagnostic procedure to determine the local CN at each of the 558 data locations;

iv. Identify the locations with CNs < 30 (i.e. *not* exhibiting collinearity).

This resulted in 80 GWR models, generated from Minkowski distances with a rotation, $\theta$, of 0°, but with varying values of $p$. In a second set of analyses, this was extended such that rotation values were allowed to vary between 0° and 90° in steps of 10°, resulting in 800 GWR models evaluated in the same way.

All of the analyses were undertaken via the R *GWmodel* package v.2.0-4 (Lu et al, 2017b).

## 3. Results

### 3.1 Initial investigations

The global correlations between the nine predictor variables are shown in Table 1 and the global CN for the OLS regression design matrix of these nine variables is 17.6. As none of the global correlations are particularly strong together with a CN < 30, it is clear that, globally at least, there is little evidence of collinearity amongst the predictors and that collinearity is highly unlikely to be a problem for this dataset.

Table 1. Global correlation coefficients between the predictor variables.

| | **TotNum** | **EdYears** | **SepBed** | **GFA** | **PlotArea** | **Frontage** | **TimeCity** | **Ppsqm** | **DistCity** |
|---|---|---|---|---|---|---|---|---|---|
| **TotNum** | 1 | -0.10 | 0.17 | 0.19 | 0.17 | 0.11 | -0.04 | 0 | 0.03 |
| **EdYears** | -0.10 | 1 | 0.06 | 0.01 | -0.11 | -0.12 | -0.07 | 0.04 | -0.12 |
| **SepBed** | 0.17 | 0.06 | 1 | 0.53 | 0.39 | 0.08 | 0.10 | -0.06 | 0.17 |
| **GFA** | 0.19 | 0.01 | 0.53 | 1 | 0.37 | 0.15 | 0.06 | -0.19 | 0.13 |
| **PlotArea** | 0.17 | -0.11 | 0.39 | 0.37 | 1 | 0.47 | 0.18 | 0.03 | 0.31 |
| **Frontage** | 0.11 | -0.12 | 0.08 | 0.15 | 0.47 | 1 | 0.08 | 0.04 | 0.20 |
| **TimeCity** | -0.04 | -0.07 | 0.1 | 0.06 | 0.18 | 0.08 | 1 | -0.06 | 0.45 |
| **Ppsqm** | 0 | 0.04 | -0.06 | -0.19 | 0.03 | 0.04 | -0.06 | 1 | -0.09 |
| **DistCity** | 0.03 | -0.12 | 0.17 | 0.13 | 0.31 | 0.20 | 0.45 | -0.09 | 1 |

An initial OLS regression analysis was undertaken to generate a model of house price, and the results are shown in Table 2. The R-squared was 0.35 and the adjusted R-squared was 0.34. The high intercept value suggests that much of the variance is being captured by this term. The coefficients for *TotNum*, *SepBed* and *Frontage* are not significantly different from zero. The model could be improved by transforming the initial data values, but in-depth model construction is not the purpose of this paper.

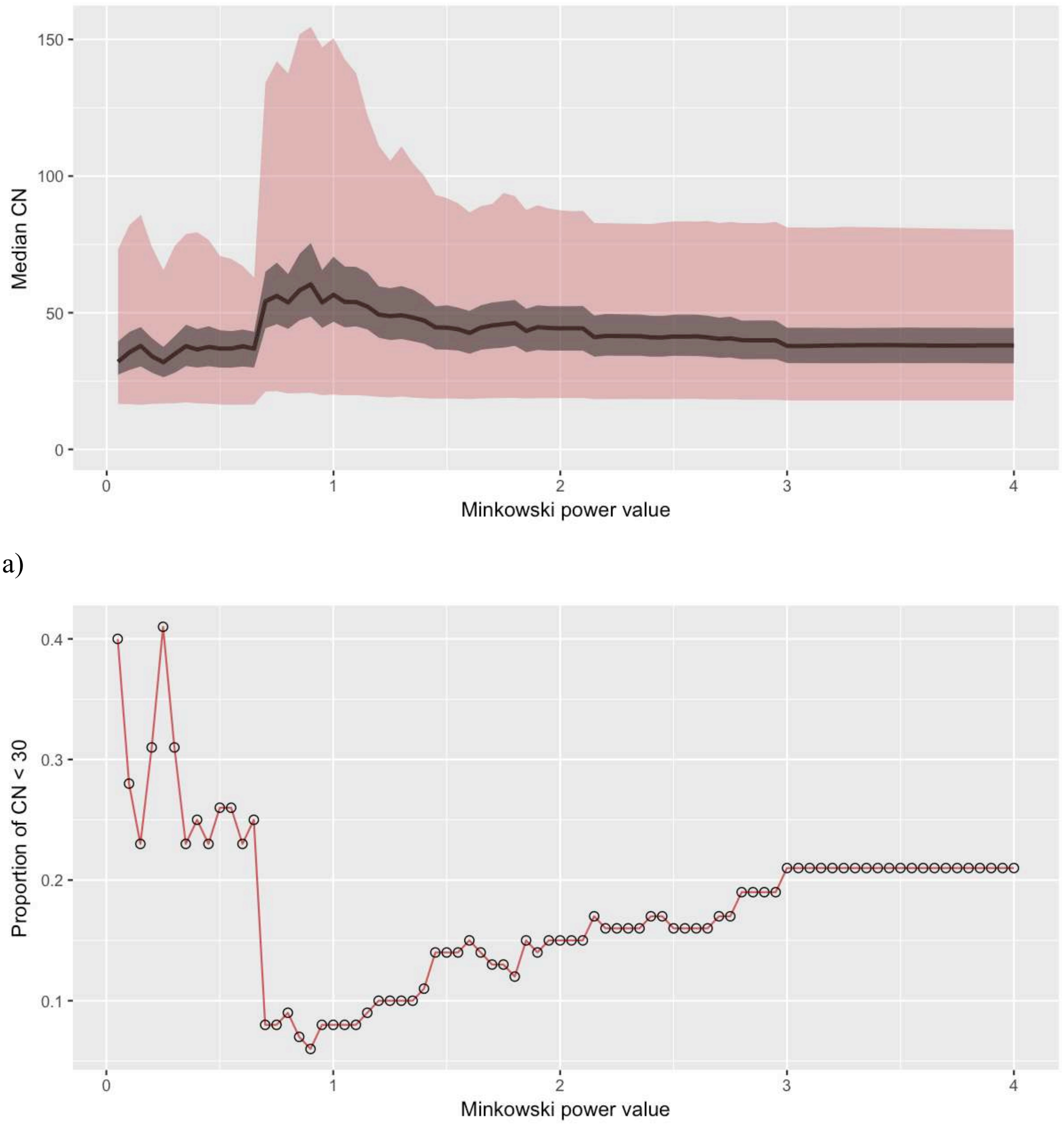
Table 2. The coefficients and associated *p*-values of the OLS regression of house price.

| | Estimate | Std. Error | t value | Pr(>\|t\|) |
|---|---|---|---|---|
| **Intercept** | 1908.678 | 966.709 | 1.974 | 0.049 |
| **TotNum** | 73.736 | 93.757 | 0.786 | 0.432 |
| **EdYears** | 148.141 | 51.385 | 2.883 | 0.004 |
| **SepBed** | -7.414 | 115.619 | -0.064 | 0.949 |
| **GFA** | 16.265 | 1.996 | 8.149 | 0.000 |
| **PlotArea** | 6.851 | 2.205 | 3.106 | 0.002 |
| **Frontage** | 35.739 | 43.503 | 0.822 | 0.412 |
| **TimeCity** | -41.486 | 12.196 | -3.402 | 0.001 |
| **Ppsqm** | 19.600 | 1.840 | 10.650 | 0.000 |
| **DistCity** | -0.367 | 0.080 | -4.619 | 0.000 |

### 3.2 Analysis: distance metric choice and collinearity

A sequence of 80 Minkowski power values were created from 0.05 to 4.00 in steps of 0.05, with a rotation, $\theta$, of 0°. For each of these, Minkowski distance matrices were created using the projected coordinates of the 558 data locations. Then, the optimal adaptive bandwidth was determined for each GWR model calibrated using each of the 80 Minkowski distance matrices, and a GWR model was constructed using that bandwidth and that Minkowski distance matrix. This resulted in 80 GWR models. For each GWR model, the local CNs for each of the 558 data locations were determined and those with a CN < 30 were identified.

Figure 3a plots all the CN values arising from each of the 80 GWR calibrations using the 80 different Minkowski distances against the distribution of CNs for that distance. Highlighted are the median CNs. In Figure 3b, the proportion of the local CNs < 30 are plotted. Here is it evident that local collinearity is a problem for a GWR fit to the case study data, but that the nature of the problem is dependent on distance metric choice. Figure 3 shows that, the proportion of local regression design matrices exhibiting local collinearity is lower for GWR fits with Minkowski distances with values of $p$ ranging from approximately *0.05* to *0.75*. It is relatively high for GWR fits with Minkowski distances with values of $p$ ranging from approximately *0.75* to *3.0*, which includes both the Euclidean distance ($p = 2$) and the Manhattan distance ($p = 1$).



a)

b)

Figure 3. The change in local CN values with changes in Minkowski distance power values, a) showing the median CN, with the CN range (red / light shade) and the CN inter-quartile range (grey / dark shade), and b) the proportion of the local regressions where CN < 30, i.e. the proportion of each GWR fit *not exhibiting* local collinearity.

One of the important advantages of Minkowski distances is their ability to handle anisotropy in distances or direction-dependent variations in distance, through the rotation parameter $\theta$. This is an important consideration in the context of GWR. Undertaking GWR in the normal way, using Euclidean distances is to assume that space is isotropic, and that the phenomenon or process under investigation decays with distance evenly regardless of direction, noting that network distances present a different case. As such, each local regression of GWR is constructed at a location and nearby observations at any given distance are weighted equally regardless of direction. This may be an unreasonable assumption where space is not isotropic and the direction in which distance is measured is important. For example, Páez (2004) reported that GWR models calibrated with anisotropic kernels outperformed those calibrated with standard isotropic distances. However, identifying the optimal anisotropic distance metric for any particular spatial process can be difficult because of the diversity within the data and the nature of the geography of the locations being considered (Lu et al. 2016). Thus the range of potentially useful distance metrics for spatial analyses is greater than a simple Euclidean or network distance.

Considering the case study, it is evident that a high degree of anisotropy might be expected as the study area contains a number of large water bodies (Figure 1). Euclidean or even network distances, commonly used in spatial analyses, may not adequately describe the directionality of the distance decay of house price. Houses immediately facing the river might be expected to have high price and those in a nearby street not facing the river, to have lower values. Similarly, houses on both sides of the river may be at a long road network distance from each other and yet be more similar in value than with houses a short distance away, but not on the river front.

Thus the next stage of the analysis was to evaluate the impact of varying the Minkowski distance rotation angle from 10° to 90° in steps of 10°, as well as varying with the power values as before. Note that the 0° case is presented in Figure 3. Figures 4 and 5 summarise the local CNs arising from each combination of Minkowski power and rotation. These results are presented in the same format as that used in Figure 3.

The results summarised by rotation angle are shown in Figure 4. These indicate that for this dataset and this set of GWR models, instances of local collinearity are highest with a rotation around 30°, regardless of the distance metric: compare for instance with Manhattan ($p = 1$), Euclidean ($p = 2$) and Minkowski ($p = 3$). However, each specific Minkowski power value has a particular minimum (or sometimes minimums), as shown in Figure 5. There are some general trends: the variation in CNs and the proportion of the local regressions of a given GWR model exhibiting collinearity are similar for Minkowski powers between 1.5 and 2.5 – i.e. approximating to Euclidean distance; rotations of around 30° have the highest instances of collinearity for Minkowski powers 0.25 to 1.00. Also of note is a peak in instances of local collinearity for a rotation angle of around 80°, with $p = 0.25$.

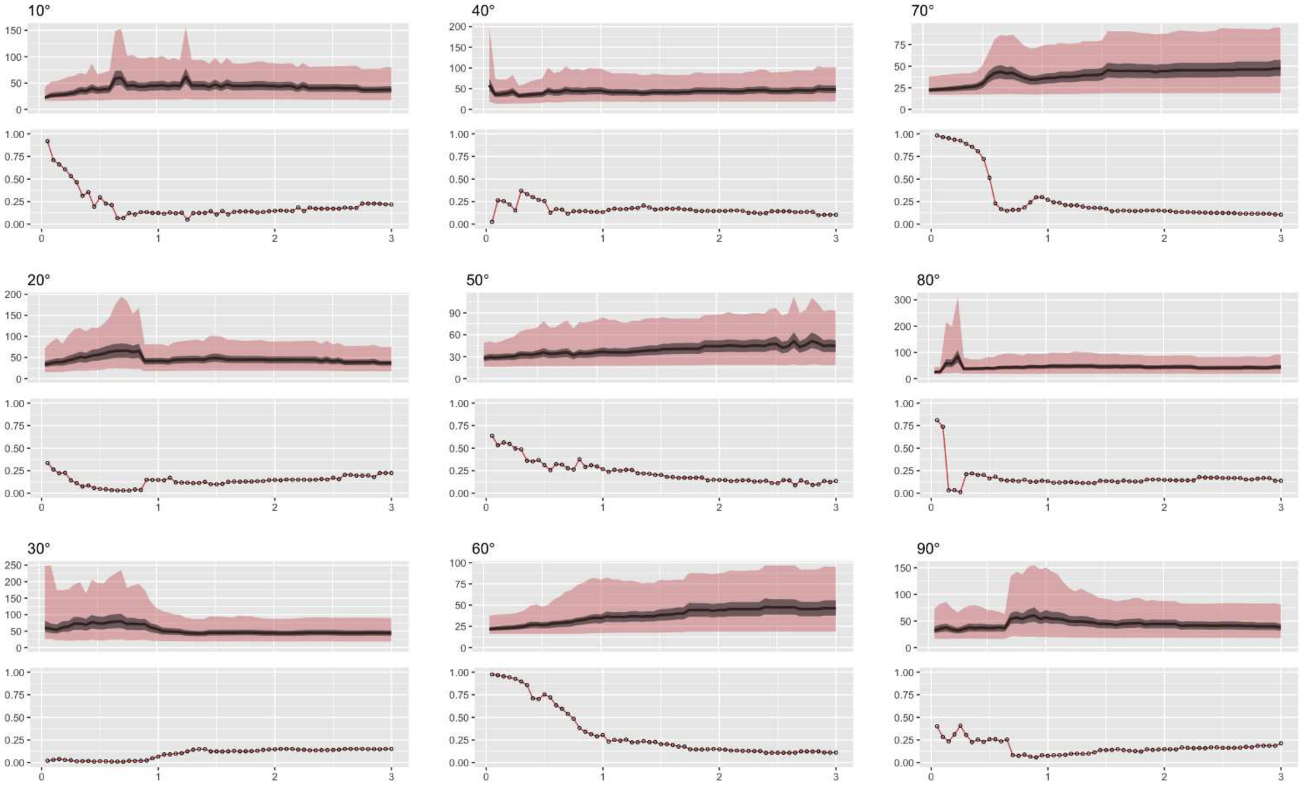
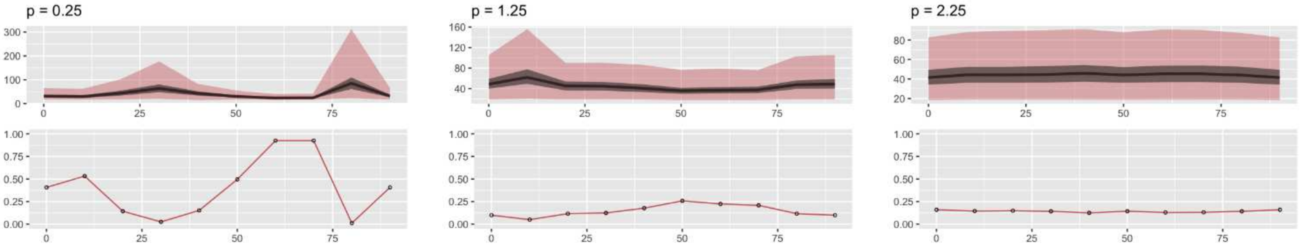


Figure 4. The changes in local CN with Minkowski power values for rotations from 10° to 90 degrees. The x-axes indicate the Minkowski power (from 0 to 3). The upper panels show the distributions of local CN values (where the median, range and IQR are highlighted) and the lower panels indicate the proportion of local regressions for each GWR fit with local CN < 30 (i.e. *not exhibiting* local collinearity).



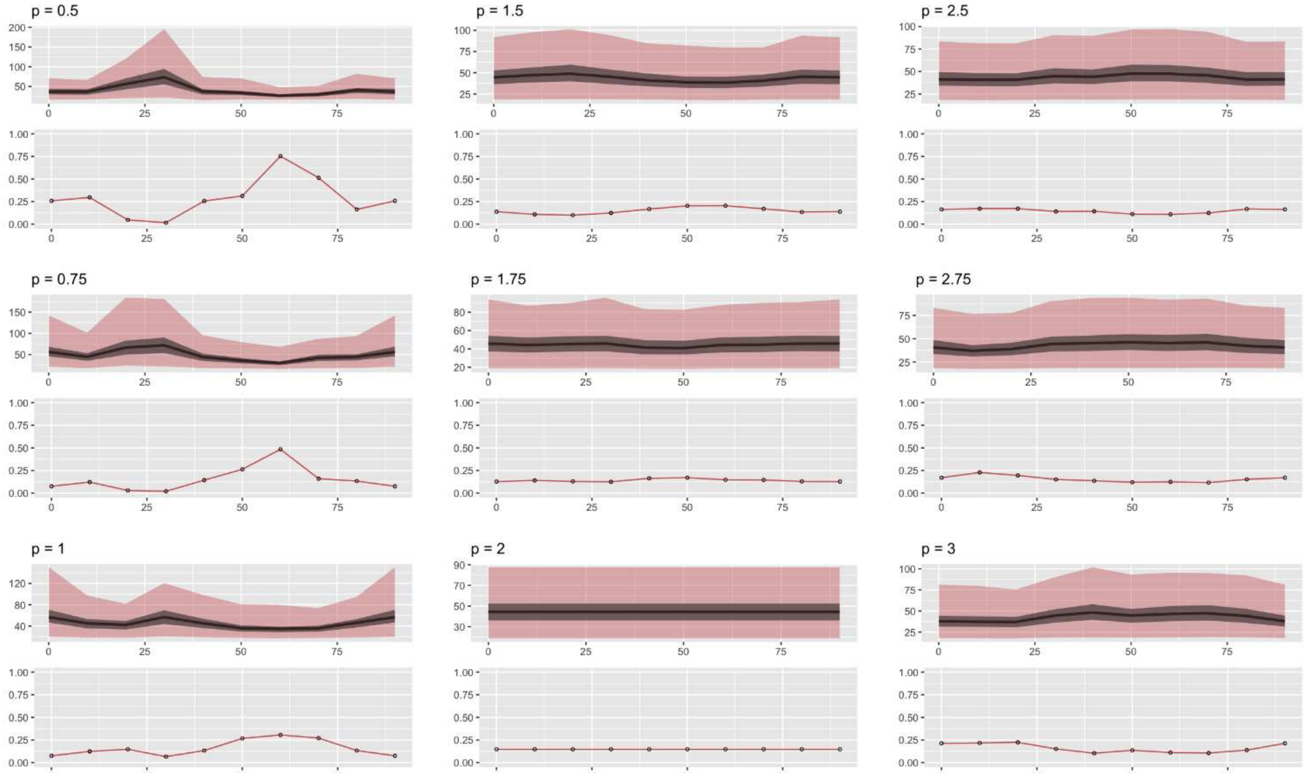


Figure 5. The changes in local CN with Minkowski rotation for power values from 0.25 to 3.0. The x-axes indicate the Minkowski rotation (from 0° to 90°). The upper panels show the distributions of local CN values (where the median, range and IQR are highlighted) and the lower panels indicate the proportion of local regressions for each GWR fit with local CN < 30 (i.e. *not exhibiting* local collinearity).

Figure 6 maps the surface of the proportions of local regressions of each GWR model with local CN < 30 (i.e. *not exhibiting* local collinearity), given by rotation angle values against Minkowski power values. GWR fits with the highest instances of local regression collinearity are found for Minkowski distances with rotations of 30° for values of *p* ranging from 0 to 0.95 and, as noted before, for a rotation of 80° and *p* values from 0.15 to 0.25 (the regions shaded 'white' in Figure 6). The combination of Minkowski power and rotation values that resulted in the lowest proportion (0.9%) of local CN < 30 (i.e. a GWR fit ***most likely*** to suffer from local collinearity), was for *p* = 0.70, rotation angle, $\theta$ = 30° (0.523 radians), and with an adaptive bandwidth of the nearest 32 data points (5.7%). Conversely, a GWR fit ***least likely*** to suffer from local collinearity), with 98.2% of the local regression models with a CN < 30 , was for *p* = 0.05, rotation angle, $\theta = 70$° (1.222 radians), and with an adaptive bandwidth of the nearest 176 data points (31.5%).

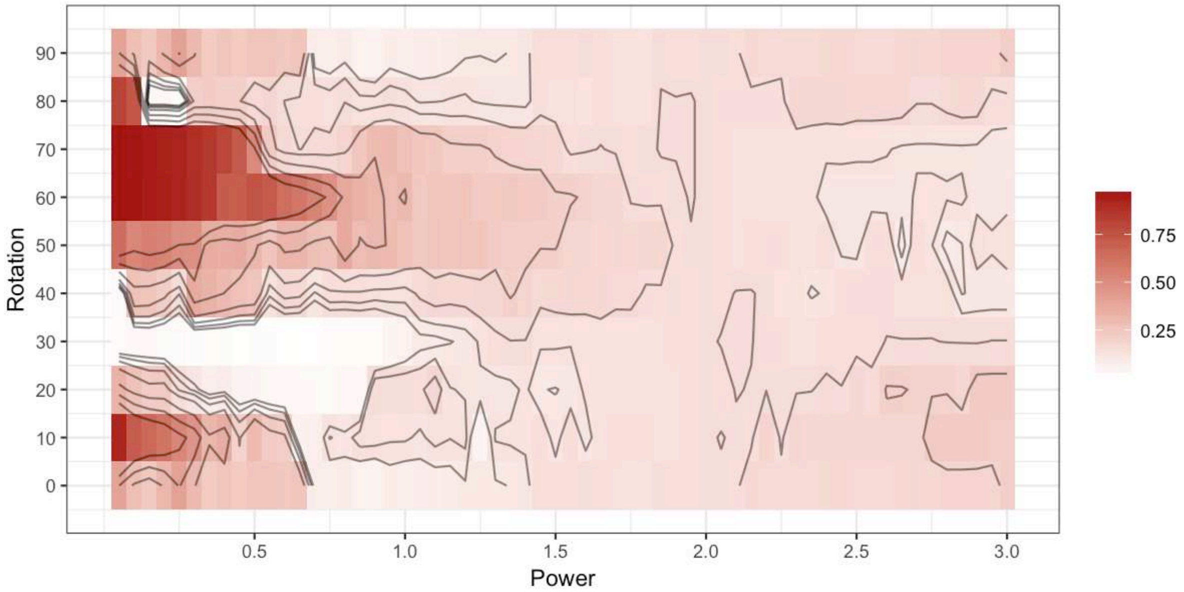


Figure 6. Surface of the proportion of local regressions of GWR fits that have local CN < 30 (i.e. *not exhibiting* local collinearity) for different combinations of Minkowski power and rotation values.

Next, as with any GWR model, the size of the bandwidth is of interest because it provides information about the expected strength of how relationships between response and predictor variables may vary across space, and in this case, with different distance metrics. Figure 7 describes a surface of the optimal adaptive bandwidths determined from different distance metrics, with different rotations and power values. The relationship between Rotation, Power and the bandwidth is very similar to that between Rotation, Power and the proportion of local models found to exhibit collinearity (Figure 6), but with some local differences as indicated in Figure 8. This plots the bandwidth against proportion of local regression models with CN < 30, for all 600 combinations of power and rotation values. There is a broadly linear relationship between bandwidth and the proportion of local models with CN < 30 – which is entirely expected as local collinearity is expected to increase as the bandwidth decreases. There is also a trend and some interaction with Power and Rotation but further investigation would be needed to unpick this.

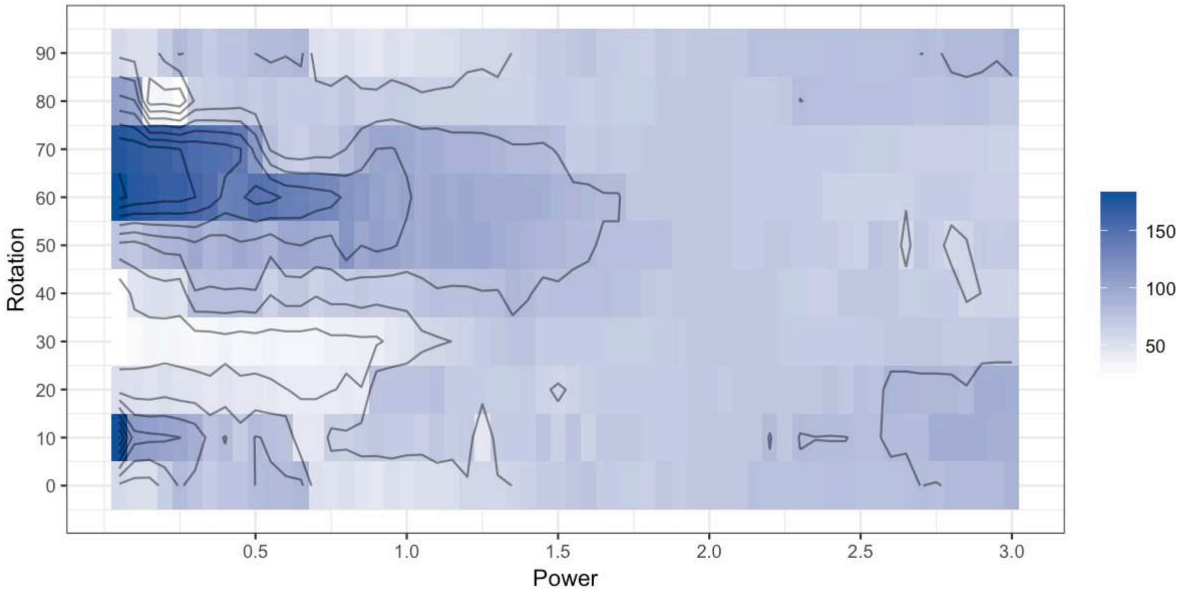


Figure 7. The surface of kernel bandwidths (given as number of nearest neighbours) for different combination of Minkowski power values and rotations.

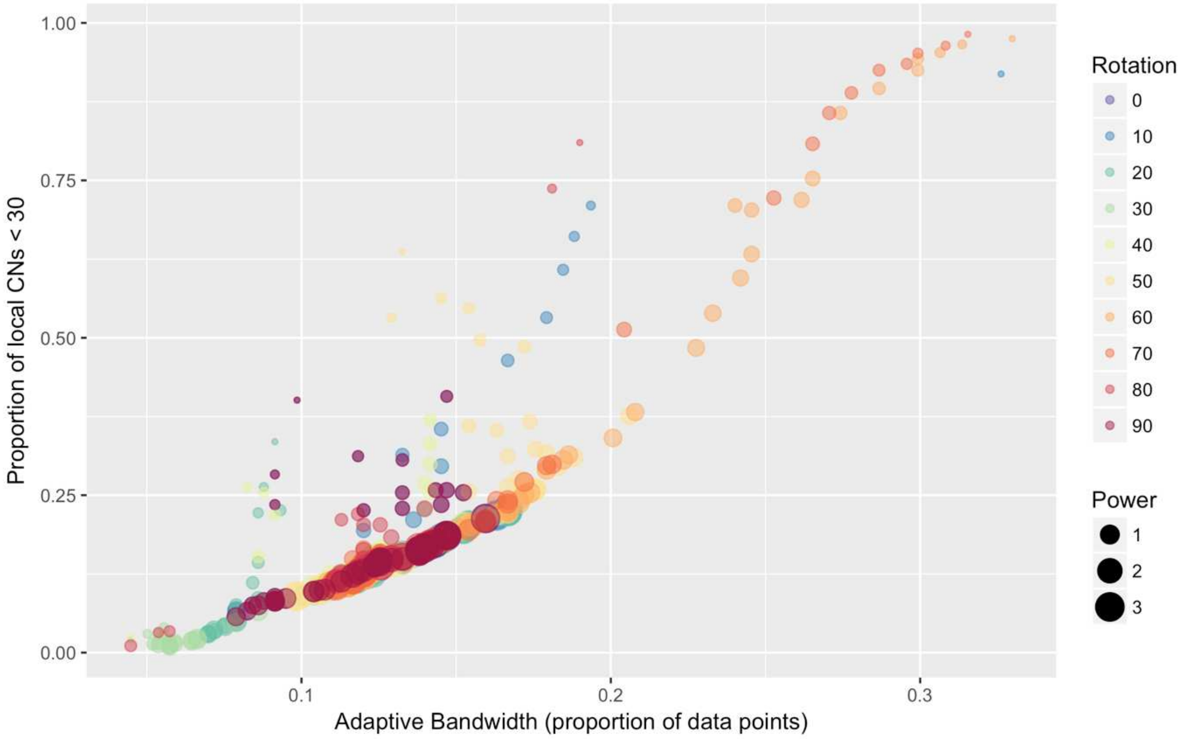
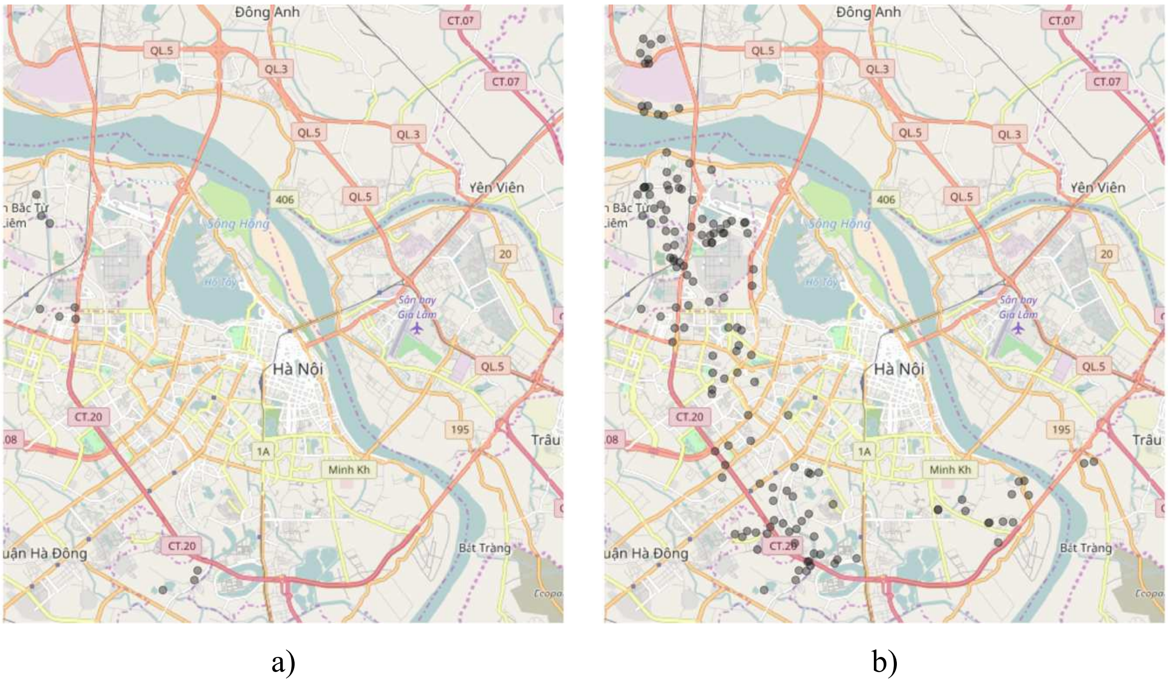
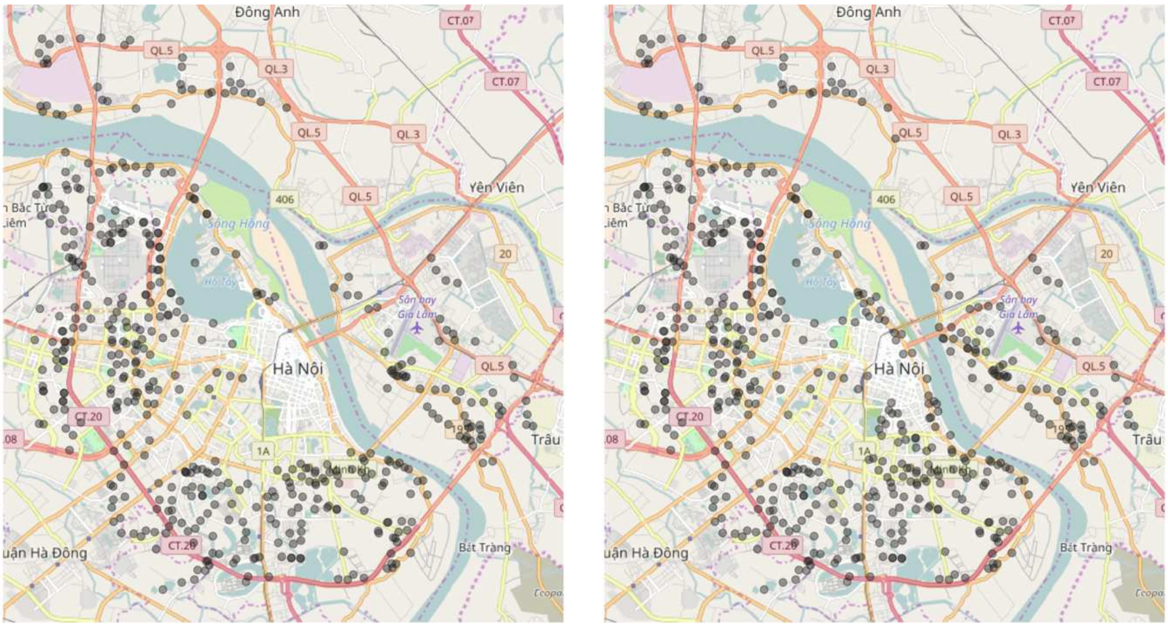


Figure 8. A plot of kernel bandwidth (given as %age) and the proportion of CNs < 30 (i.e. *not exhibiting* local collinearity) against adaptive bandwidth with Minkowski power values indicated by plot character size and Minkowski rotation values indicated by the shading.

Finally, it is important to provide a series of maps of local CN > 30, for different GWR fits (Figure 9). Here four GWR fits were chosen by varying the power and rotation of the Minkowski distance, ranging from a GWR fit least affected by local collinearity, to a GWR fit most affected by local collinearity. These GWR models can be summarised as follows:

- Minkowski $p = 0.05$, Minkowski rotation angle $\theta = 70$°, a resultant optimal adaptive bandwidth of 31.5% and local collinearity found in 1.8% of the local regressions;
- $p = 0.10$, $\theta = 80$°, bandwidth = 18.1%, 26.3% collinearity;
- $p = 1.75$, $\theta = 30$°, bandwidth = 11.8%, 87.5% collinearity;
- $p = 0.7$, $\theta = 30$°, bandwidth = 5.7%, 99.1% collinearity.





c) d)

Figure 9. A comparison of four different combinations of Minkowski power and rotation values depicting where the resultant GWR fits *exhibit* local collinearity (i.e. only plot where the local CN > 30) in: a) 1.8% (the minimum), b) 26.3%, c) 87.5%, and d) 99.1% (the maximum) of the local regression models for each of the four GWR fits.

## 4. Discussion and concluding remarks

This paper investigates the impacts of different distance metrics on collinearity in the resultant GWR models. The case study in Hà Nội, Vietnam has large water bodies throughout the city centre in the form of lakes and the Red River, suggesting distance metric choice to be an important consideration. In this context network distances, may be appropriate – they might capture the structure of water bodies in the study area – but equally may not adequately describe the distance decay of house prices, as properties immediately on opposite sides of the river may be far away by network distance from each other and yet be more similar in value than with houses a short distance away, but not on the river front. However, the aim of the paper was to not to explore the local collinearity associated with network distances, but the degree to which different Minkowski distances can possibly capture some aspects of spatial structure. Here, the Minkowski distance found to maximise local collinearity in the GWR fit was approximate to a Manhattan distance ($p = 0.7$) but also included a rotation of 30°. The Minkowski distance found to minimise local collinearity in the GWR fit was parametrised with $p = 0.05$ and a rotation of 70°.

Collinearity in regression models can result in loss of precision and power in the coefficient estimates. Regression analyses operate under the assumption of independence in predictor variables. If these are correlated, then the regression model can be sensitive to random errors in the response variable which can result in a large variance thereby reducing model inferential power. Collinearity can be a particular problem in local approaches such as GWR, because the predictor variables in the data subsets may exhibit collinearity locally even when none is observed globally (Wheeler and Tiefelsdorf, 2005). A number of approaches have been suggested for handling this in GWR (Wheeler 2007, 2009; Brunsdon et al 2012) transferring concepts in regression modelling more generally (Hoerl 1962, Hoerl and Kennard 1970; Tibshirani 1996; Zou and Hastie 2005). These solutions have typically taken one of two routes: altering the estimator to include a small change to the values of the diagonal of the cross-product matrix, referred to as the ridge (Hoerl 1962; Hoerl and Kennard 1970), in order to increase the difference between the diagonal and off-diagonal elements of the matrix, and in so doing reducing the collinearity among the predictors. Or to use a shrinkage estimator to generate coefficient estimates that are biased to be small in order to constrain them, with the so called lasso (Tibshirani, 1996). Elastic nets (Zou and Hastie, 2005) have also been proposed and are a hybrid of ridge regression and lasso regularization.

The results of this study's analysis suggest a further consideration for reducing local collinearity in GWR, one that incorporates some form of distance attenuation to weight data locally, providing a complementary perspective to the importance of bandwidth selection in GWR analyses. Bandwidth choice is always emphasised as being *the* critical factor in correctly parameterising a GWR model (e.g. Lu et al., 2014b; Gollini et al., 2015). It is also well-known, that bandwidth selection will directly affect collinearity, with small bandwidths more likely to result in local collinearity problems, than found with large bandwidths (Brunsdon et al. 2012). The study presented here indicates that the choice of distance metric may be just as important, as it can negatively and positively affect the collinearity within local data subsets. The most effective distance metric, of course will relate to the characteristics of the case study data, both in terms of the Minkowski power and the rotation. The methods described in this paper allow both power and rotation angle to vary fully and can be used to determine how to best parameterise GWR with a distance metric, in respect to collinearity.

Although this research highlights an important specification issue in GWR, worthy of reporting, there exists practical caveats that should be adhered to. There are critical tensions in the context of distance metric choice and addressing collinearity in GWR. Firstly, the *distance metric* is *problem-dependent*. This suggests that distance metrics should be carefully chosen to reflect knowledge of the spatial processes being investigated, to reflect notions of "closeness", that relate, for example, to the local pattern of house price variations and predictors, such that similar patterns are closer using that distance metric than dissimilar patterns. Thus, in this case study, $p = 1$ (close to the network distance) could be preferred to $p = 2$ (Euclidean distance), as it reduces local collinearity, whilst at the same time is not a distance metric that has a rather abstract meaning (e.g. $p = 0.05$, which minimised collinearity). Secondly, *collinearity* is *sample-dependent*, which for GWR, is also dependent on the bandwidth. Collinearity should still primarily be dealt with using standard procedures (e.g. ridge, lasso). However, as demonstrated, distance metric choice can influence collinearity, but its reduction via distance metrics should not be at the expense of changing the purpose and logic of the GWR model. In the extreme, a distance metric could be chosen such that all sample points are the same distance apart and collinearity minimised, but and this would simply fall back to the global OLS regression fit, so care must be exercised.

In summary, GWR is an inherently exploratory approach and understanding how the distance metric and the kernel bandwidth interact with collinearity has the capacity to provide further insight into the nature and structure of the data relationships being examined. This paper provides evidence that distance metric choice can provide a useful extra tuning component to address local collinearity, not only for basic GWR (as demonstrated), but also for adapted GWR models. Adaptations not only include those already addressing collinearity, such as the ridge or lasso, but also others such as robust (for outliers) and heteroskedastic (for error variance) forms (Fotheringham et al. 2002) – GWR models that may not be so easily adapted to ridge or lasso forms. Brunsdon et al. (2012) have already observed that bandwidth size and collinearity strongly interact, and allowed GWR bandwidths to be locally-specified so that local collinearity is reduced (i.e. specify bandwidths such that all local CNs < 30). This study has now highlighted a further important interaction with respect to distance metric choice.

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