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ON THE INTRICACY OF COMBINATORIAL CONSTRUCTION PROBLEMS

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Preface

A combinatorial research week was held at the Open University from 20–23 September 1982, at which D.E. Daykin described how he and R. Häggkvist had conceived a concept of 'intricacy', and posed the problem [5] of showing that the intricacy of latin squares is two. This inspired the participants to develop the concept and apply it to a variety of combinatorial problems, both during the week and in collaboration thereafter.

The original latin square problem remains in precisely the same state of partial solution as was achieved by Daykin and Häggkvist; but the concept which it generated seems to be of enough inherent interest to be worthy of exposition.

The name W.E. Opencomb is a flag of convenience under which Open University combinatorial weeks aspire to sail. The participants on this occasion were: R.A. Bailey, P.J. Cameron, A.G. Chetwynd, D.E. Daykin, A.J.W. Hilton, F.C. Holroyd, J.H. Mason, R. Nelson, C.A. Rowley and D.R. Woodall.

1. Introduction

Some combinatorial objects can readily be constructed by 'greedy algorithms'. For example, a spanning tree of a connected finite graph can be constructed an edge at a time, merely by checking at each step that a circuit is not formed. This process never fails to construct a spanning tree, and every spanning tree can be thus constructed. On the other hand, if we try to construct an $n \times n$ latin square by filling in the entries one by one, checking at each stage that no entries in that row or column have been filled with the same symbol, we will frequently come to a halt before a latin square has been constructed. We say that the spanning tree problem is *simple*, and the latin square problem *intricate*, its *intricacy* being the smallest integer k such that a 'failed' partial $n \times n$ latin square can always be partitioned into k or fewer parts, each of which can be extended to an $n \times n$ latin

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square. (We shall see in Section 5 that for $n>1$ the intricacy of the latin square problem is always between 2 and 4.)

In this paper we give general definitions of construction problems, intricacy and related concepts, and explore these ideas in a variety of particular cases.

2. Construction problems and their intricacy

A *construction problem* $\mathscr{C}$ is a system $(D, \mathscr{P}, \mathscr{G})$, where:

(i) D is a finite non-empty set, the *domain* of the problem;

(ii) $\mathscr{P}$, the set of *partial structures*, is a subset of 2^D containing every singleton;

(iii) $\mathscr{G}$, the set of *goal structures*, is a non-empty set of maximal elements of $\mathscr{P}$ (considering $\mathscr{P}$ to be ordered by inclusion);

(iv) $\mathscr{P}$ is hereditary; that is, every subset of a partial structure is a partial structure.

The hereditary property implies that every partial structure may be obtained as the final element of a chain $P_1 \subset P_2 \subset \cdots \subset P_k$ where each P_i is of cardinality i. Those maximal elements of $\mathscr{P}$ that are not goal structures are called *failures*. The partial structures that are subsets of goal structures are called *extensible*; thus a partial structure is non-extensible if and only if every maximal structure containing it is a failure. If there are no failures (or equivalently, if every partial structure is extensible), then $\mathscr{C}$ is *simple*; otherwise it is *intricate*.

Example 1. The spanning tree problem mentioned in the Introduction is a simple construction problem; D is the edge set of the graph in question, the partial structures are the acyclic subgraphs, and the goal structures are the spanning trees.

Example 2. The $n\times n$ latin square problem mentioned in the Introduction may be posed as a construction problem as follows:

$$\mathscr{L}_n = (B\times C, \mathscr{P}, \mathscr{G}), \qquad n>1,$$

where:

(i) B is the set of cells of an $n\times n$ matrix;

(ii) C is a set of n symbols;

(iii) $\mathscr{P}$ is the set of all subsets of $B\times C$ of the form $\{(b, f(b)) \colon b\in S\}$, where S is a subset of B, and $f(b_1)\neq f(b_2)$ whenever b_1, b_2 are distinct and in the same row or column;

(iv) $\mathscr{G}$ is the subset of $\mathscr{P}$ for which $S=B$.

A construction problem satisfying (iii) and (iv) for some sets B and C is called a *colouring problem*. (C is the set of colours, B is the set of objects being coloured.) It is frequently more convenient to define it by partial functions and functions from B to C rather than by the corresponding subsets of $B\times C$. Thus, $\mathscr{L}_n$ may be identified as $(B \times C, \Pi, \Gamma)$ where Π and Γ are the appropriate sets of partial functions and functions respectively.

$\mathscr{L}_n$ is intricate for $n > 1$, since the partial latin square

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 1 & \\ & & & & & 2 \end{bmatrix}$$

is always a failure.

Given any subset $\mathscr{G}$ of 2^D, it is in principle always possible to define a simple problem with $\mathscr{G}$ as the set of goal structures, by defining $\mathscr{P}$ to be the set of all subsets of goal structures. However, it seems clear that in order for $(D, \mathscr{P}, \mathscr{G})$ to represent a plausible or interesting construction problem, the partial structures must be easy to recognize inductively. That is to say, given a partial structure of cardinality k, it must be easy to determine which elements of D can be adjoined to it to produce partial structures of cardinality $k+1$.

A problem $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$ is *fair* if every singleton of D is extensible. (The $n \times n$ latin square problem is fair; the problem whose partial structures are partial matchings and whose goal structures are 1-factors is unfair if not every edge of G belongs to a 1-factor.) Given a fair problem $\mathscr{C}$, its *intricacy* $\kappa(\mathscr{C})$ is the smallest positive integer i such that every partial structure can be partitioned into i or fewer extensible structures. Clearly $\kappa(\mathscr{C}) = 1$ if and only if $\mathscr{C}$ is simple, and any simple problem is fair. According to our previous definition, an unfair problem is necessarily intricate; but we do not assign a numerical value to the intricacy of such a problem.

Remark 1. The hereditary nature of partial structures implies that $\kappa(\mathscr{C})$ can equivalently be defined as the smallest i such that every *failure* is the *union* (not necessarily disjoint) of at most i extensible structures.

Let $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$ be a construction problem. A subset S of D is *free* if $P \cap S$ is extensible for every partial structure P (or equivalently, if every subset of S which is a partial structure is extensible). A *template* for $\mathscr{C}$ is a partition of D into free sets. The *freedom*, $\varphi(\mathscr{C})$, of $\mathscr{C}$ is the largest integer i such that every subset of D of cardinality i is free; the *template number*, $\tau(\mathscr{C})$, of a fair problem is the smallest cardinality of a template. Clearly, $\mathscr{C}$ is fair if and only if $\varphi(\mathscr{C}) \geqslant 1$.

Remark 2. Any partition of D into extensible structures is a template, but a template need not be of this form.

A *uniform* problem is one whose goal structures are precisely the partial structures of a given cardinality $\nu(\mathscr{C})$, the *value* of $\mathscr{C}$; thus all the other partial structures are of lesser cardinality. All problems considered in this paper except those in Section 8 are uniform. In a uniform problem $\mathscr{C}$, the cardinality of a failure cannot exceed $\nu(\mathscr{C})-1$. In certain cases (see Lemma 7 in Section 9, for example) the maximum cardinality of a failure is less than this. The maximum cardinality $\omega(\mathscr{C})$ of any failure of $\mathscr{C}$ is called the *watershed* of $\mathscr{C}$.

Finally, a partial structure of $\mathscr{C}$ is *decadent* if its only extensible subsets are singletons; the *decadence* of $\mathscr{C}$, $\delta(\mathscr{C})$, is the maximum cardinality of a decadent partial structure.

Where no confusion can arise, we abbreviate $\kappa(\mathscr{C})$ to κ, etc.

Theorem 1. *Let $\mathscr{C}$ be a fair construction problem. Then:*

(i) *one of δ and φ is equal to* 1 *and the other exceeds* 1 (*unless all the goal structures are singletons, in which case* $\varphi=\delta=1$);

(ii) $\delta \leqslant \kappa \leqslant \tau$;

(iii) $\kappa \leqslant \lceil \omega/\varphi \rceil$;

(iv) *if $\mathscr{C}$ is uniform, then* $\tau \leqslant \min(|D|-\nu+1,\ \lceil |D|/\varphi \rceil)$.

Proof. (i) Since $\mathscr{C}$ is fair, $\varphi \geqslant 1$. If $\varphi > 1$, then every two-element subset of every partial structure is extensible, so $\delta=1$. If $\varphi=1$, then either all the goal structures are singletons, in which case $\delta=1$, or there is at least one non-extensible (and hence decadent) two-element partial structure, so $\delta \geqslant 2$.

(ii) These inequalities follow directly from the definitions.

(iii) This follows directly from Remark 1.

(iv) Let G be any goal structure. Since $\mathscr{C}$ is fair, the partition of D into G and $(|D|-\nu)$ singletons is a template. Also, D can be partitioned into $\lceil |D|/\varphi \rceil$ subsets each of cardinality at most φ; this is another template, so the result follows. □

Corollary 1.1. *If a decadent structure and a template of the same cardinality k can be found for a problem, then $\delta=\kappa=\tau=k$.*

Proof. This follows directly from part (ii) of the theorem. □

A problem obeying the conditions of Corollary 1.1 is said to be *k-regular*. We shall see in the course of this paper that there exist both regular and non-regular problems.

If all the domain elements of a problem $\mathscr{C}$ are interchangeable, the problem is said to be *transitive*. More precisely, let Aut $\mathscr{C}$ be the permutation group on D

whose induced action on 2^D leaves the sets $\mathcal{P}$ and $\mathcal{G}$ invariant; then $\mathcal{C}$ is transitive if Aut $\mathcal{C}$ acts transitively (i.e., given a, b in D there exists π in Aut $\mathcal{C}$ such that $\pi(a)=b$). The inequality of Theorem 1(iv) can be supplemented in the case of transitive problems, by an argument attributed by L. Babai to Lovasz [4, Remark 5.5].

Theorem 2. *Let $\mathcal{C}$ be uniform and transitive. Then*

$$\tau \leqslant \left\lfloor \frac{\log |D|}{\log |D| - \log(|D| - \nu)} \right\rfloor + 1 \leqslant \left\lfloor \frac{|D| \log |D|}{\nu} \right\rfloor + 1.$$

Proof. Let G be any goal structure, k any positive integer, and select at random k translates of G under Aut $\mathcal{C}$. Then for any element d of D, transitivity implies that the probability of d not belonging to any of these translates is $(1-\nu/|D|)^k$. Thus the probability that the union of the translates is not the whole of D is bounded above by $|D|\,(1-\nu/|D|)^k$. This becomes less than 1 when

$$k > \frac{\log |D|}{\log(|D| - \log(|D| - \nu)};$$

hence when this inequality holds there must be some choice of k translates of G whose union is D; say $G_1, \ldots, G_k$. Setting $S_i = G_i \setminus \bigcup_{j=1}^{i-1} G_j$, the set $\{S_i\}_{i=1,\ldots,k}$ is a template, and the first inequality is established.

Now for any $y \geqslant 1$, $\log y \geqslant 1 - 1/y$. Setting $y = |D|/(|D| - \nu)$ shows that

$$\frac{\log |D|}{\log |D| - \log(|D| - \nu)} \leqslant \frac{|D| \log |D|}{\nu},$$

which proves the second inequality. □

3. An example: covering rectangular boards with dominoes

Consider the problem of covering a $p \times q$ rectangular board (i.e., p rows and q columns of unit squares) with non-overlapping $1 \times r$ dominoes. We denote the squares by integer pairs (i, j) $(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q)$, from top to bottom and left to right. We denote a typical horizontal domino position by $H(i, j)$, representing a domino covering the squares $(i, j), (i, j+1), \ldots, (i, j+r-1)$, and a typical vertical position by $V(i, j)$, representing a domino covering the squares $(i, j), (i+1, j), \ldots, (i+r-1, j)$. The range of permissible i, j is, of course, determined by p, q and r.

Lemma 1. *A $p \times q$ board can be completely covered with non-overlapping $1 \times r$ dominoes if and only if r divides p or q.*

Proof. The condition is obviously sufficient. To see that it is neceseary, let $c_1, \ldots, c_r$ be a set of r colours, and colour the squares of the board as follows: (i, j) receives the colour c_k where $k \equiv i+j (\mathrm{mod}\, r)$. Then any domino position covers exactly one square of each colour; but it is not difficult to see that, unless r divides p or q, there are unequal numbers of squares in the various colour classes. □

If p, q, r are positive integers such that $r \geqslant 2$ and r divides q, we define the construction problem $\mathcal{D}_{p,q,r}$ as follows: the domain is the set of all permissible $1 \times r$ domino positions on a $p \times q$ board, the partial structures are the sets of non-overlapping domino positions, and the goal structures are the partial structures that completely cover the board.

Lemma 2. *If $p < r$ and $q = mr$, then $\mathcal{D}_{p,q,r}$ is simple if $m = 1$ and unfair otherwise.*

Proof. There is only one goal structure, and if $m > 1$ there are domino positions that do not belong to it. □

Thus we may restrict our serious attention to the problems $\mathcal{D}_{r+n,mr,r}$ where $n \geqslant 0$, $m \geqslant 1$ and $r \geqslant 2$.

Lemma 3. *If $n \geqslant 0$, $m \geqslant 1$ and $r \geqslant 2$, then $\mathcal{D}_{r+n,mr,r}$ is a fair problem.*

Proof. Consider first any vertical domino position $V(i, j)$. The set of all vertical domino positions with row number i is a partial structure which is clearly extensible by means of horizontal domino positions. Thus the singleton $\{V(i, j)\}$ is extensible.

Next, consider any horizontal domino position $H(i, j)$. Select an integer k such that $1 \leqslant k \leqslant i \leqslant k+r-1 \leqslant r+n$. Let $\mathcal{K}_1 = \{h(l, j): k \leqslant l \leqslant k+r-1\}$, and let $\mathcal{K}_2$ be the set $\{V(k, s): 1 \leqslant s < j \text{ or } j+r \leqslant s \leqslant mr\}$. Then $\mathcal{K}_1 \cup \mathcal{K}_2$ is a partial structure containing $H(i, j)$, which is extensible by means of horizontal domino positions (Fig. 1). Thus the singleton $\{H(i, j)\}$ is extensible. □

Theorem 3. *$\mathcal{D}_{r+n,r,r}$ is a k-regular problem with $k = \min(n+1, r)$ for all $r \geqslant 2$, $n \geqslant 0$.*

Proof. Let $k = \min(n+1, r)$. Consider the set $\{V(i, i): 1 \leqslant i \leqslant k\}$ of non-overlapping domino positions (Fig. 2).

This is a decadent set, since if $i < j \leqslant k$, the presence of dominoes $V(i, i)$ and $V(j, j)$ prevents the square $(j-1, j)$ from being covered by any domino. Thus,

$$\delta(\mathcal{D}_{r+n,r,r}) \geqslant k. \tag{1}$$

Now let $\mathcal{H}$ be the set of all horizontal domino positions, and for each

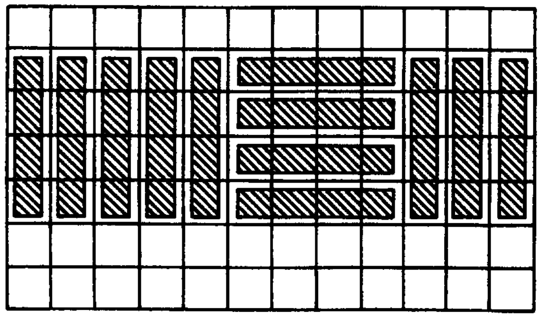

Fig. 1.

$i = 1, \ldots, k$ let $\mathcal{V}_i$ be the set of all vertical domino positions whose row number is congruent to $i (\operatorname{mod} r)$. Then the partition

$$\{\mathcal{H} \cup \mathcal{V}_i, \mathcal{V}_2, \ldots, \mathcal{V}_k\}$$

of all domino positions is a template for $\mathcal{D}_{r+n,r,r}$. Thus,

$$\tau(\mathcal{D}_{r+n,r,r}) \leqslant k, \tag{2}$$

and the result follows from (1), (2) and Corollary 1.1. □

Theorem 4. *$\mathcal{D}_{r+n,mr,r}$ is an $(r+n)$-regular problem for all $r \geqslant 2$, $m > 1$, and n in the range $0 \leqslant n < r$.*

Proof. Let $\mathcal{H}$ be the set of domino positions $\{H(i,k)\colon 1 \leqslant i \leqslant r+n,\ 2 \leqslant k \leqslant r\}$, and let $H(i,k) \in \mathcal{H}$. In any goal structure containing $H(i,k)$, the squares $(i,1), (i,2), \ldots, (i,k-1)$ must be covered by vertical dominoes $V(l,1), V(l,2), \ldots, V(l,k-1)$ for some constant l, and the squares above and below these vertical dominoes must be covered with horizontal dominoes. (Fig. 3.) Thus two distinct elements $H(i_1,k_1)$ and $H(i_2,k_2)$ of $\mathcal{H}$ cannot belong to the same goal structure unless $1 \leqslant |i_1 - i_2| \leqslant r-1$ and $k_1 = k_2$. Moreover, any goal structure containing the domino position $H(r,1)$ must contain all $H(i,1)$ $(1 \leqslant i \leqslant r+n)$, and cannot therefore contain any element of $\mathcal{H}$. It follows that the set

$$\{H(1,r), H(2,r-1), \ldots, H(r,1), H(r+1,r),$$
$$H(r+2,r-1), \ldots, H(r+n,r-n+1)\},$$

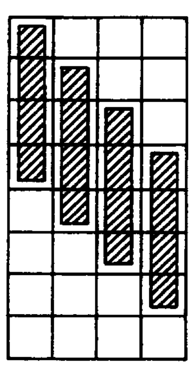

Fig. 2.

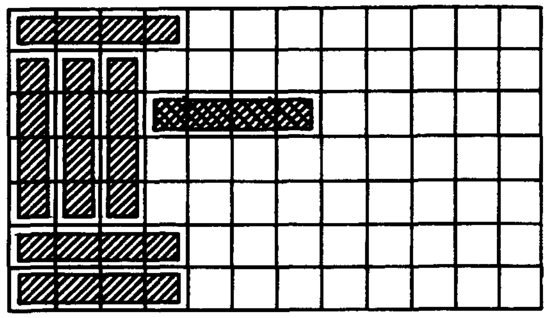

Fig. 3.

depicted in Fig. 4, is decadent. Thus,

$$\delta(\mathscr{D}_{r+n,mr,r}) \geq r+n \tag{3}$$

in the range considered.

Now, for each $i=1,\ldots,r+n$, let $\mathscr{S}_i$ be the set of all domino positions which are either horizontal in row i or (where they exist) vertical with the top square in row i. It is straightforward to verify that for each i, any non-overlapping subset of $\mathscr{S}_i$ can be completed to a goal structure. Thus $\{\mathscr{S}_i: 1 \leq i \leq r+n\}$ is a template, and

$$\tau(\mathscr{D}_{r+n,mr,r}) \leq r+n \tag{4}$$

in the range considered. The result follows from (3), (4) and Theorem 1. □

Theorem 5. (i) $\delta(\mathscr{D}_{r+n,mr,r}) \geq r$ *for all* $r \geq 2$, $m>1$ *and* $n \geq r$.

(ii) $\tau(\mathscr{D}_{lr,mr,r}) \leq 2r-2$ *for all* $r \geq 2$ *and* l, $m>1$.

(iii) $\tau(\mathscr{D}_{lr+n,mr,r}) \leq 2r-1$ *for all* $r \geq 2$, l, $m>1$ *and* $0<n<r$.

Proof. (i) The set $\{(i, r-i+1): 1 \leq i \leq r\}$ is decadent since, if $i<j$, the presence of dominoes $H(i, r-i+1)$ and $H(j, r-j+1)$ in a partial structure prevents the square $(i, r-i)$ from being covered by any domino in that structure.

(ii), (iii) For each integer i between 1 and r inclusive, we define sets $\mathscr{I}_i$, $\mathscr{U}_i$ of domino positions, as follows. Partition the board into $r \times r$ squares, and rectangles, by cutting along all lines separating row numbers congruent to $i-1$ from row numbers congruent to i, and column numbers congruent to i from column numbers congruent to $i+1$ (all modulo r). Fig. 5 illustrates the case $r=4$, $n=1$,

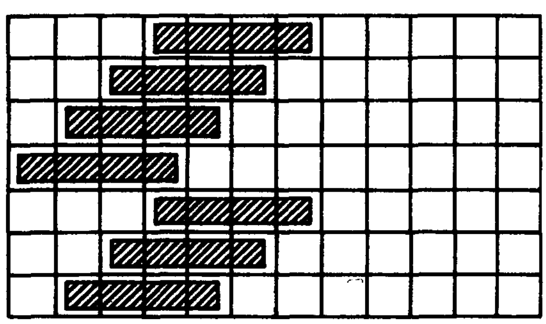

Fig. 4.

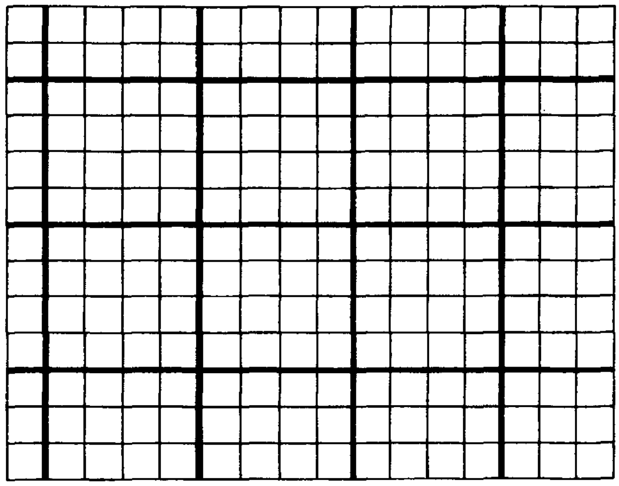

Fig. 5.

$l=3$, $m=4$, $i=3$. Then $\mathcal{J}_i$ consists of all the $2r$ domino positions lying entirely within an $r \times r$ square, together with all vertical domino positions lying entirely within any of the rectangles of height r which exist at either end of the board unless $i=r$. $\mathcal{U}_i$ consists of all horizontal domino positions lying entirely within any of the rectangles of width r and height less than r which always exist at the top and/or bottom of the board unless $n=0$ and $i=1$. If $n=0$, then $\mathcal{U}_1$ is defined to be empty.

Now in any partial structure which is a subset of $\mathcal{J}_i$, no $r \times r$ square can contain both a horizontal and a vertical domino. Thus it is not difficult to see that any partial structure which is a subset of any of $\mathcal{J}_1, \mathcal{U}_1, \mathcal{J}_2, \mathcal{U}_2, \ldots, \mathcal{U}_{r-1}, \mathcal{J}_r \cup \mathcal{U}_r$ is extensible, and hence each of these sets is free. Taken together they form a partition of the domain. Moreover, $\mathcal{U}_1$ is empty when $n=0$. Thus in each case we have a template of the required cardinality. $\square$

Specializing to the case $r=2$, Theorem 3 to 5 establish precisely the decadence, intricacy and template numbers of $\mathcal{D}_{2l,2m,2}$, $\mathcal{D}_{2l+1,2,2}$, and $\mathcal{D}_{3,2m,2}$; but when l, $m>1$ all that the above theorems say concerning $\mathcal{D}_{2l+1,2m,2}$ is that each of these numbers is 2 or 3. The next theorem completes the specification in this case.

Lemma 4. *Let l, $m>1$. Then a non-overlapping pair P of domino positions of $\mathcal{D}_{2l+1,2m,2}$ is extensible unless it isolates a corner square from the remainder of the board.*

Proof. Let P be a non-overlapping pair which does not isolate a corner square. Then it may be straightforwardly verified that P fulfils at least one of the three following conditions:

(1) The board can be split vertically into two boards each containing an even number of columns, with one element of P in each board.

(2) The board can be split horizontally into two boards each containing at least two rows, and there is one element of P on each board.

(3) P may be extended to a covering of a 4×4 square by eight non-overlapping dominoes.
If P fulfils conditions (1) or (2), then P is extensible by Lemma 3. If it fulfils condition (3), it is extensible by elementary considerations. □

Theorem 6. *Let l, $m>1$. Then:*
(i) $\delta(\mathscr{D}_{2l+1,2m,2})=2$;
(ii) $\kappa(\mathscr{D}_{2l+1,2m,2})=\tau(\mathscr{D}_{2l+1,2m,2})=3$.

Proof. (1) By Lemma 4, no set of three non-overlapping dominoes is decadent; but there exist decadent pairs, namely those which isolate a corner square.

(ii) Consider the partial structure $\mathscr{L}=\{H(1,2),\ H(2,1),\ H(3,2),\ H(4,1),\dots,H(2l,1),\ H(2l+1,2)\}$ (Fig. 6). We shall show that this set cannot be partitioned into two extensible structures.

Suppose $\{\mathscr{M},\mathscr{N}\}$ is a partitioning of $\mathscr{L}$ into two extensible structures. Then one of $H(1,2)$ and $H(2,1)$ must belong to $\mathscr{M}$ and one to $\mathscr{N}$. Suppose that one of $H(2i-1,2)$ and $H(2i,1)$ belongs to $\mathscr{M}$ and one to $\mathscr{N}$ for each $i=1,\dots,j$, but that $H(2j+1,2)$ and $H(2j+2,1)$ are both in the same set, say $\mathscr{M}$. Let $\mathscr{G}$ be any goal structure containing $\mathscr{M}$. In order that the square $(2j+1,1)$ be covered, we must have $V(2j,1)\in\mathscr{G}$. It then follows that $V(2j-2,1)\in\mathscr{G}$, then $V(2j-4,1)\in\mathscr{G},\dots,V(2,1)\in\mathscr{G}$. But we then have the contradiction that the square $(1,1)$ is not covered by a domino of $\mathscr{G}$. Thus one of $H(2i-1,2)$ and $H(2i,1)$ belongs to $\mathscr{M}$ and one to $\mathscr{N}$ for all $i=1,\dots,l$.

By a similar argument, starting from the bottom of the board rather than the top, we also find that one of $H(2i+1,2)$ and $H(2i,1)$ belongs to $\mathscr{M}$ and one to $\mathscr{N}$ for all $i=1,\dots,l$. Thus the members of $\mathscr{M}$ and $\mathscr{N}$ alternate down the board. But this contradicts the fact that the set $\{H(1,2),H(3,2),\dots,H(2l+1,2)\}$ cannot be extensible, owing to the impossibility of covering column 1.

Thus $\mathscr{L}$ cannot be partitioned into two extensible structures, and $\kappa\geqslant 3$. The fact that $\kappa=\tau=3$ now follows from Theorems 5 and 1(ii). □

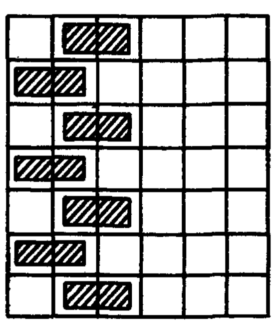

Fig. 6.

4. Some relationships between construction problems

There are a number of interesting construction problems which can be expressed in graph-theoretic terms as the construction of: 1-factors, 1-factorizations, edge-colourings, hamiltonian circuits, hamiltonian decompositions, maximum flows, and decompositions into triangles. The description of these problems can be facilitated, and certain relationships between them revealed, by means of two operators, 'Pack' and 'Col', which act on construction problems.

The 'Pack' *operator.* Given a construction problem $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$, its *packing number*, $p(\mathscr{C})$, is the largest number of mutually disjoint goal structures in $\mathscr{G}$. The problem 'Pack $\mathscr{C}$' has $\mathscr{G}$ as its domain; its partial structures are the sets of mutually disjoint elements of $\mathscr{G}$; and its goal structures are the partial structures of cardinality $p(\mathscr{C})$.

The Col *operator.* Give a construction problem $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$, its *colouring number*, $c(\mathscr{C})$, is the smallest number c such that D can be partitioned into c partial structures. Let C be the set $\{1, 2, \ldots, c\}$; then the colouring problem 'Col $\mathscr{C}$' has $D \times C$ as its domain; the goal structures are the functions f from D to C such that $f^{-1}(i)$ belongs to $\mathscr{P}$ for each i in C; and the partial structures are the partial functions with this property.

It sometimes happens that $p(\mathscr{C}) = c(\mathscr{C})$ and the sets of $p(\mathscr{C})$ mutually disjoint goal structures are exactly the partitions of D into $c(\mathscr{C})$ partial structures. In this case, the goal structures of Pack $\mathscr{C}$ and Col $\mathscr{C}$ can be thought of as "essentially the same," although the partial structures, which one can think of intuitively as determining the method of construction, will be quite different. There are two instances of this in Example 3 below, and Example 4 provides a further instance.

Example 3. Let G be any finite simple non-null graph. Let $\mathscr{E}(G)$ be the problem of constructing an edge: the domain is the vertex set, the goal structures are the vertex pairs representing edges, and the partial structures are the singletons and the goal structures. (Thus $\mathscr{E}(G)$ is simple unless G possesses isolated vertices, in which case it is unfair.) The use of the above operators then generates the following construction problems.

(a) Pack $\mathscr{E}(G)$ is the problem of finding a maximum matching in G, by selecting an edge at a time.

(b) Col $\mathscr{E}(G)$ is the problem of colouring the vertices of G, one at a time, using the minimum number of colours, such that each colour class is either a single vertex or an adjacent pair.

If G possesses a 1-factor, then $p(\mathscr{E}(G)) = c(\mathscr{E}(G))$ and each of these problems is essentially that of constructing a 1-factor, but by different methods.

(c) Col Pack $\mathscr{E}(G)$ is the problem of finding a proper edge-colouring of G with the minimum number of colours, by colouring an edge at a time.

(d) $\text{Pack}^2\, \mathscr{E}(G)$ is the problem of finding a maximum set of mutually disjoint maximum partial matchings, by selecting these one at a time. If G possesses a 1-factorization, then $p(\text{Pack}\, \mathscr{E}(G)) = c(\text{Pack}\, \mathscr{E}(G))$ and each of the two problems above is essentially that of constructing a 1-factorization, but by different methods.

Example 4. Let G be a graph with an edge-decomposition into hamiltonian circuits, and let $\mathcal{H}(G)$ be a construction problem whose domain is the set of edges of G and whose goal structures are the hamiltonian circuits. (The nature of its partial structures is irrelevant at present.) Then G is a regular graph, $p(\mathcal{H}(G))$ and $c(\mathcal{H}(H))$ are each equal to half the vertex degree, and Pack $\mathcal{H}(G)$ and Col $\mathcal{H}(G)$ are problems involving finding hamiltonian decompositions of G.

Example 5. If in a network with integer capacities each edge of capacity k is replaced by k edges of unit capacity, then the problem of finding a maximum flow between two given points v, w can be interpreted as the problem 'Pack $\mathcal{X}$' where the goal structures $\mathcal{X}$ are the vw-paths. The interpretation is equally valid if we have directed flows and directed paths.

Example 6. The problem of finding a Steiner triple system on n points (where $n \equiv 1$ or 3 (mod 6)) can be interpreted as 'Pack $\mathcal{J}_n$' where the goal structures of $\mathcal{J}_n$ are the sets of edges of the complete graph K_n that form triangles.

Theorem 7. *For any construction problem* $\mathcal{C} = (D, \mathcal{P}, \mathcal{G})$,

$$\kappa(\mathrm{Col}\,\mathcal{C}) \leq c(\mathcal{C}).$$

Proof. Let C be the set $\{1, 2, \ldots, c\}$ and let π be any c-cycle on C and g any goal structure (i.e., any function from D to C obeying the appropriate conditions). The definition of Col $\mathcal{C}$ implies that the elements $1, 2, \ldots, c$ are interchangeable, and hence for each $i = 1, \ldots, c-1$, the function $\pi^i g$ is a goal structure. Thus each of the subsets of $D \times C$ corresponding to the functions g, $\pi g, \ldots, \pi^{c-1} g$ is free, and these subsets form a template of cardinality c. Applying Theorem 1(ii), we have $\kappa(\mathrm{Col}\,\mathcal{C}) \leq \tau(\mathrm{Col}\,\mathcal{C}) \leq c$. □

We now consider three ways of producing what might be considered as a 'smaller' problem from a given construction problem $\mathcal{C} = (D, \mathcal{P}, \mathcal{G})$. We call these *restriction*, *sharpening* and *concentration*, and they involve passing to a proper subset of D, $\mathcal{P}$ and $\mathcal{G}$ respectively.

Restriction. Let $\mathcal{C} = (D, \mathcal{P}, \mathcal{G})$ be a construction problem, and let E be a proper subset of D. Denote $\{P \cap E : P \in \mathcal{P}\}$ by $\mathcal{P}_E$ and $\{G \cap E : G \in \mathcal{G}\}$ by $\mathcal{G}_E$. Then the system $\mathcal{D} = (E, \mathcal{P}_E, \mathcal{G}_E)$ is itself a construction problem provided that every element of $\mathcal{G}_E$ is maximal in $\mathcal{P}_E$. $\mathcal{D}$ is then said to be a *restriction* of $\mathcal{C}$.

Example 7. Any proper edge-colouring of K_{2m-1} with $2m-1$ colours has a different colour missing at each vertex, and is thus a restriction of a colouring of K_{2m} with $2m-1$ colours. Hence Col Pack $\mathcal{E}(K_{2m-1})$ is a restriction of Col Pack $\mathcal{E}(K_{2m})$.

Any problem $\mathscr{C}$ has a fair restriction, which we denote by Fair $\mathscr{C}$, obtained by restricting the domain to the set of all extensible singletons; thus Fair $\mathscr{C} = \mathscr{C}$ if and only if $\mathscr{C}$ is fair.

Sharpenings. Let $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$, and let $\mathscr{Q}$ be a proper subset of $\mathscr{P}$ which still contains all of $\mathscr{G}$, and still has the hereditary property. Then the construction problem $(D, \mathscr{Q}, \mathscr{G})$ is said to be a *sharpening* of $\mathscr{C}$. A sharpening corresponds to being 'cleverer' in one's construction attempts.

Example 8. Consider again the problem $\mathscr{L}_n$. If $\frac{1}{2}n < m < n$, then clearly no $n \times n$ latin square on n symbols can contain an $m \times m$ latin square on m symbols as a submatrix. Thus we may sharpen $\mathscr{L}_n$ by means of a new partial structure set $\mathscr{Q}$ which excludes all partial latin squares containing such submatrices.

Concentrations. Let $\mathscr{C} = (D, \mathscr{P}, \mathscr{G})$, and let $\mathscr{H}$ be a proper subset of $\mathscr{G}$; then the problem $(D, \mathscr{P}, \mathscr{H})$ is said to be a *concentration* of $\mathscr{C}$. For example, we can form a concentration of $\mathscr{L}_n$ by retaining the same set of partial structures, but requiring the goal structures to be those latin squares that can be interpreted as Cayley tables for groups. (We shall formulate this more precisely in Section 8.)

The following theorem follows immediately from the definitions.

Theorem 8. *The intricacy of a fair construction problem is not increased by sharpening or restricting the problem, and is not decreased by concentrating it (assuming the concentration to be a fair problem).*

5. Matchings, edge-colourings and 1-factorizations

Example 3 of Section 4 described five construction problems associated with a graph G. In the present section we consider three of these, namely Pack $\mathscr{E}(G)$, Col Pack $\mathscr{E}(G)$ and $\text{Pack}^2\,\mathscr{E}(G)$, for certain particular graphs.

Concerning Pack $\mathscr{E}(G)$, we remark that a simple problem is frequently obtained: for example, when G is regular of degree 1, and when G is a complete graph or a complete bipartite graph. On the other hand, the following theorem shows that the intricacy may be arbitrarily high.

Theorem 9. *For each positive integer n let H_n be the graph obtained by joining the corresponding odd vertices of a pair of paths of $2n-1$ vertices each* (Fig. 7). *Then* $\kappa(\text{Pack}\,\mathscr{E}(H_n)) \geq n$.

Proof. Any edge joining one of the paths to the other (drawn horizontally in Fig. 7) uniquely determines a 1-factor, and this 1-factor contains no other such edges.

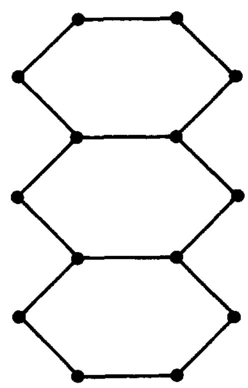

Fig. 7.

Thus the set of all these edges is decadent. The result now follows from Theorem 1(ii). □

Theorem 10. (i) $\kappa(\text{Col Pack}\,\mathscr{C}(K_n))=1$ *if* $n=2$ *or* 3.
(ii) $2\leq\kappa(\text{Col Pack}\,\mathscr{C}(K_n))\leq 6$ *if* $n>3$.

Proof. Col Pack $\mathscr{C}(K_2)$ and Col Pack $\mathscr{C}(K_3)$ are clearly simple, and for $n>3$ it is easy to find a proper partial edge-colouring of K_n that cannot be completed. (For example, if n is even, choose a 1-factor, and colour all but one of its edges one colour and the last edge a different colour. If n is odd, take a K_{n-1}-subgraph of K_n and edge-colour it using $n-2$ colours.) Therefore $\kappa(\text{Col Pack}\,\mathscr{C}(K_n))\geq 2$ if $n>3$.

Now by Example 7 and Theorem 8, all that remains is to show that $\kappa(\text{Col Pack}\,\mathscr{C}(K_{2m}))\leq 6$ for $m>1$. The proof splits into two cases.

Case (a): $m=2q$ *for some integer* q. Partition the vertices of K_{4q} into four sets $V_1,\ldots,V_4$ each containing q vertices. For $1\leq j<k\leq 4$ let I_{jk} be the induced subgraph on $V_j\cup V_k$. The edge-set of K_{4q} is the union of those of the graphs I_{jk}. The restriction of any partial structure to I_{jk} can (by a 'greedy algorithm') be extended to a proper edge-colouring of I_{jk} using up to $2m-1$ colours. By [1, Corollary 4.3.3], each of these can then be extended to a proper edge-colouring of K_{2m}. Thus each partial structure is the union of six extensible structures, and the result follows from Remark 1.

Case (b): $m=2q+1$ *for some integer* q. Consider any failure f, and let x and y be two vertices of K_{4q+2} such that the edge joining them is uncoloured by f. The restrictions of f to the induced subgraphs on $\{x\}\cup V_1\cup V_2$, $\{x\}\cup V_3\cup V_4$, $V_1\cup V_3$, $V_2\cup V_4$, $\{y\}\cup V_1\cup V_4$, $\{y\}\cup V_2\cup V_3$ can each be extended to proper edge-colourings of these complete graphs, and thus to proper edge-colourings of K_{2m}, by the same argument as in Case (a). Thus each failure is the union of six extensible structures, and the result follows from Remark 1. □

Conjecture 1. $\kappa(\text{Col Pack}\,\mathscr{C}(K_n))=2$ for all $n>3$.

Next, we consider the edge-colouring problem for complete bipartite graphs; or equivalently, the latin square problem.

Theorem 11. *For $n>1$, $2 \leqslant \kappa(\mathscr{L}_n) \leqslant 4$, and $\varphi(\mathscr{L}_n)=n-1$.*

Proof. We have already observed (in Section 2) that $\mathscr{L}_n$ is intricate if $n>1$. The proof that $\kappa(\mathscr{L}_n) \leqslant 4$ splits into two cases; in each of these, we denote by X and Y the sets of vertices such that the edges all join vertices in X to vertices in Y.

Case (a): $n=2m$ *for some integer* m. Partition X into V_1 and V_2, and Y into V_3 and V_4, such that each V_j contains m vertices. For $j=1,2$ and $k=3,4$ let I_{jk} be the induced subgraph on $V_j \cup V_k$. Any partial structure, restricted to I_{jk}, can be extended by a greedy algorithm to a proper edge-colouring of I_{jk}, and hence by Ryser's Theorem [12] to a proper edge-colouring of $K_{n,n}$. Thus $\kappa(\mathscr{L}_{2m}) \leqslant 4$.

Case (b): $n=2m+1$ *for some integer* m. Let f be any failure. Select vertices x in X, y in Y, joined by an edge that is uncoloured by f. Partition the remainder of X into V_1 and V_2, and the remainder of Y into V_3 and V_4, such that each V_j contains m vertices. Then the argument of Case (a) applies to the restriction of f to the induced subgraphs on each of $\{x\} \cup V_1 \cup V_3$, $\{y\} \cup V_2 \cup V_3$, $\{y\} \cup V_1 \cup V_4$, $\{x\} \cup V_2 \cup V_4$. Thus $\kappa(\mathscr{L}_{2m+1}) \leqslant 4$.

The argument showing that $\mathscr{L}_n$ is intricate also shows that $\varphi(\mathscr{L}_n)<n$, while Evan's conjecture (proved by Andersen and Hilton [3], and by Smetaniuk [13]) is a statement that $\varphi(\mathscr{L}_n) \geqslant n-1$. □

Conjecture 2. $\kappa(\mathscr{L}_n)=2$ for each $n>1$.

It is of interest to ask what is the range of possible values for the intricacy of the edge-colouring problem for a graph of chromatic index c. Theorem 7 shows that it cannot exceed c, and the star graphs and odd cycle graphs illustrate that it can be as low as 1. The following theorem shows that the value c can always be obtained.

Theorem 12. *For every positive integer c, there exists a graph M_c such that* Col Pack $\mathscr{E}(M_c)$ *is a regular problem, with intricacy c.*

Proof. Let M_c be the graph formed from two copies of the complete bipartite graph $K_{c,c-1}$, by joining the $(c-1)$-valent vertices in the first copy to those in the second copy by a set of vertex-disjoint edges $e_1, \ldots, e_c$ (Fig. 8).

Since M_c is bipartite, it has chromatic index c by König's Theorem [7, Theorem 4.3]. However, by counting colours at the c-valent and $(c-1)$-valent vertices of each $K_{c,c-1}$, it is clear that each edge e_j must receive a different colour in any proper edge-colouring with c colours. Thus the partial structure consisting of $e_1, \ldots, e_c$ all with the same colour is decadent, so $c \leqslant \delta$. Since Theorem 7 establishes the existence of a template of cardinality c, we have $\tau \leqslant c$, and the result now follows from Theorem 1(ii). □

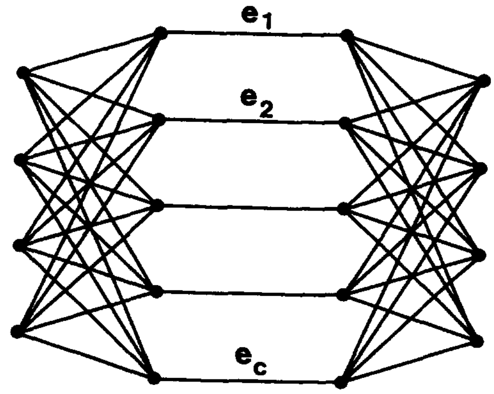

Fig. 8.

Consider next the problem $\mathrm{Pack}^2\,\mathscr{E}(K_{2m})$; this is the problem of constructing an edge-colouring (or equivalently a 1-factorization) of K_{2m} by selecting 1-factors one at a time.

Theorem 13. $\kappa(\mathrm{Pack}^2\,\mathscr{E}(K_{2m})) \leqslant \lceil \frac{1}{3}(2m-3) \rceil$ *for* $m \geqslant 4$.

Proof. It follows from [11, Corollary 3.5] that $\varphi(\mathrm{Pack}^2\,\mathscr{E}(K_{2m})) \geqslant 3$ for $m \geqslant 4$, and the result then follows directly from Theorem 1(iii), since clearly $\omega(\mathrm{Pack}^2\,\mathscr{E}(K_{2m})) \leqslant 2m-3$. □

6. Hamiltonian circuit and hamiltonian decomposition problems

If G is a graph with at least one hamiltonian circuit, we define $\mathscr{H}(G)$ to be the construction problem $(D, \mathscr{P}, \mathscr{G})$ where D is the edge-set of G, $\mathscr{G}$ the set of hamiltonian circuits of G, and $\mathscr{P}$ the union of $\mathscr{G}$ with the set of *path systems* of G, i.e., acyclic subgraphs of maximum valency at most 2. (The temptation to 'sharpen' $\mathscr{H}(G)$ by replacing the set of path systems by the set of single paths must be resisted; the resulting set of partial structures is not hereditary.)

Clearly if G is a complete graph or the complete bipartite graph $K_{n,n}$, then $\mathscr{H}(G)$ is simple. On the other hand, the intricacy may be arbitrarily high, as the following theorem shows.

Theorem 14. *For each positive integer n let F_n be the union of $K_{n,n+1}$ with a path through the $n+1$ vertices of degree n of $K_{n,n+1}$* (Fig. 9). *Then* $\kappa(\mathscr{H}(F_n)) \geqslant n$.

Proof. Every hamiltonian circuit of F_n uses exactly one edge of the extra path. Hence the partial structure consisting of the edges of the path is decadent. The result now follows from Theorem 1(ii). □

Next, we consider the hamiltonian decomposition problem $\mathrm{Col}\,\mathscr{H}(K_{2n+1})$. The following lemma is a slight rephrasing of [8, Theorem 2].

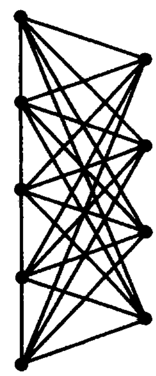

Fig. 9.

Lemma 5. *Let $1 \leq r < 2m+1$. Then an edge-colouring of K_r with up to m colours, each colour class being a path system, can be extended to a hamiltonian decomposition of K_{2m+1} if and only if for each colour there are at least $2(r-m)-1$ edges of that colour in K_r.*

Theorem 15. *For $m > 1$, $2 \leq \kappa(\text{Col}\,\mathcal{H}(K_{2m+1})) \leq 6$.*

Proof. For any $m > 1$, if we take a hamiltonian circuit of K_{2m+1} and colour all but one of its edges one colour and the last edge a different colour, then the result cannot extend to a hamiltonian decomposition. Thus $\kappa(\text{Col}\,\mathcal{H}(K_{2m+1})) \geq 2$.

The proof that $\kappa(\text{Col}\,\mathcal{H}(K_{2m+1})) \leq 6$ splits into two cases.

Case (a): $m = 2q$ *for some integer q*. Let f be any partial colouring, x any vertex of K_{2m+1}. Partition the remainder of the vertices into four sets $V_1, \ldots, V_4$ (each containing q vertices) in such a way that for every pair of similarly-coloured edges incident to x, one of the corresponding vertices lies in $V_1 \cup V_2$ and the other in $V_3 \cup V_4$. Consider the restriction of f to the induced subgraphs on each of $\{x\} \cup V_1 \cup V_2$, $\{x\} \cup V_3 \cup V_4$, $V_1 \cup V_3$, $V_1 \cup V_4$, $V_2 \cup V_3$, $V_2 \cup V_4$. Each of these can be extended by a 'greedy algorithm' to a colouring of the corresponding complete subgraph in which each colour class is a path system. Moreover, in the first two induced subgraphs, the colours incident to x given by f are all distinct, so that by colouring the remaining edges incident to x first we may ensure that all m colours are used. Thus by Lemma 5 each of these partial colourings extends to a hamiltonian decomposition of K_{2m+1}.

Case (b): $m = 2q+1$ *for some integer q*. Let f be any failed structure. Let x and y be two vertices such that the edge joining them is uncoloured by f. Partition the remaining vertices into $V_1, \ldots, V_4$ such that $|V_1| = q+1$, $|V_2| = |V_3| = |V_4| = q$, and for every pair of similarly-coloured edges incident to x [resp y], one of the corresponding vertices lies in $V_1 \cup V_2$ [resp $V_1 \cup V_3$] and the other in $V_3 \cup V_4$ [resp $V_2 \cup V_4$]. Consider the restriction of f to the induced subgraphs on each of the following vertex sets:

$$\{x\} \cup V_1 \cup V_2, \{y\} \cup V_1 \cup V_3, V_1 \cup V_4,$$
$$V_2 \cup V_3, \{y\} \cup V_2 \cup V_4, \{x\} \cup V_3 \cup V_4.$$

The same argument as in Case (a) applies, and the result follows. □

7. Steiner triple systems

For any set X and any positive integer k not exceeding $|X|$, we denote by $\binom{X}{k}$ the set of subsets of X of cardinality k. A *Steiner triple system* (STS) of order $n \geqslant 3$ is a set S of cardinality n together with a subset T of $\binom{S}{3}$ such that every element of $\binom{S}{2}$ is contained in exactly one element of T. It is well-known that an STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$ [10]. In this case, we say that n is *admissible.*

If we visualize the elements of S as the vertices of the graph K_n, and we let $\mathscr{J}_n$ be a construction problem whose domain is the edge set and whose goal structures are the triangles of K_n, then we can interpret 'Pack $\mathscr{J}_n$' as the problem of constructing an STS of order n (when n is admissible).

The following lemma indicates a general technique for obtaining upper bounds on the intricacy of block design construction problems, which we then apply to the case of Steiner triple systems.

Lemma 6. *Let $\mathscr{C}$ be a construction problem with domain $\binom{S}{k}$ for some finite set S. Suppose there exist integers $p \geqslant 1$, $q \geqslant 0$ with the property that any partial structure of $\mathscr{C}$ is extensible provided that it belongs to $\binom{Y}{k}$ for some $Y \subseteq S$ with $|Y| \leqslant (|S|-q)/p$. Suppose also that there exists a finite set X and a subset $\mathscr{B}$ of $\binom{X}{b}$ (where $b < |X|/p$), such that every element of $\binom{X}{k}$ is a subset of some element of $\mathscr{B}$. Finally, suppose that*

$$|S| \geqslant \left\lceil \frac{(p-1)b+q}{|X|-pb} \right\rceil |X|.$$

Then $\kappa(\mathscr{C}) \leqslant |\mathscr{B}|$.

Proof. Let $|S| = n$ and $|X| = x$. Then n may be expressed as

$$n = rx + l \tag{5}$$

where r, l are non-negative integers, $l < x$, and

$$r \geqslant \frac{(p-1)b+q}{x-pb}. \tag{6}$$

We may re-express (6) thus:

$$rx + b \geqslant p(br+b)+q. \tag{7}$$

Now let $m = br + \min(l, b)$. In the case $l \geqslant b$, expressions (5) and (7) immediately yield the inequality

$$n \geqslant pm + q. \tag{8}$$

In the case $l<b$, we have

$$\begin{aligned} n &= (rx+b)-(b-l) \\ &\geqslant p(br+b)+q-(b-l) \\ &\geqslant p(br+l)+q \qquad (\text{since } p\geqslant 1) \end{aligned}$$

and we again obtain expression (8).

Now partition $\mathcal{S}$ into x subsets $S_1, \ldots, S_x$ of which the first l have cardinality $r+1$ and the remainder have cardinality r. We may take the set X described in the lemma to be $\{S_1, \ldots, S_x\}$. For every element B of $\mathcal{B}$, the union of the elements of B is a subset of S of cardinality at most m, which we denote by B^*. Let P be any partial structure of $\mathcal{C}$. Then the set of all elements of P that belong to $\binom{B^*}{k}$ is extensible; but every element of P belongs to $\binom{B^*}{k}$ for some B in $\mathcal{B}$, and the result follows. □

Theorem 16. *If* $n\geqslant 260$ *and* $n\equiv 1$ *or* $3 \pmod 6$, *then* $2\leqslant \kappa(\mathrm{Pack}\, \mathcal{J}_n)\leqslant 130$.

Proof. In any STS of order at least 7, we can always find triangles $\{a, b, c\}$, $\{c, d, e\}$, $\{e, f, a\}$, such that a, b, c, d, e, f are distinct. If we remove these triangles from the system and add the triangle $\{a, c, e\}$, we obtain a partial structure of Pack $\mathcal{J}_n$ which is a failure. Thus $\kappa(\mathrm{Pack}\, \mathcal{J}_n)\geqslant 2$.

Now by [2, Theorem 1], all partial structures of Pack $\mathcal{J}_m$ are extensible structures of Pack $\mathcal{J}_n$ whenever $n\geqslant 4m+1$. Moreover, the inversive plane $E(2, 5)$ (see [6, Section 2.4]) is a subset of $\binom{X}{6}$ of cardinality 130, where $|X|=26$, such that every triple of X is contained in an element of $E(2, 5)$. The result now follows from Lemma 6, with $p=4$, $q=1$, $\mathcal{B}=E(2, 5)$, $b=6$, $k=3$. □

If $n\equiv 3$ or $7 \pmod{12}$, this bound can be considerably improved.

Theorem 17. $\kappa(\mathrm{Pack}\, \mathcal{J}_n)\leqslant 20$ *whenever* $n\equiv 3$ *or* 7 $(\mathrm{mod}\ 12)$.

Proof. Let $n\equiv 3$ or $7 \pmod{12}$. Set $n=4m+3$; then n and $2m+1$ are both admissible. Let P be any partial structure of Pack $\mathcal{J}_n$; i.e. any set of edge-disjoint triangles of K_n (whose vertex set we denote by V). Select three vertices a, b, c so that $\{a, b, c\}$ is not a triangle of P. Partition $V\backslash\{a, b, c\}$ into $X\cup Y$, where $|X|=|Y|=2m$, in such a way that no triangle of P with vertex a has both of its other vertices in $X\cup\{b\}$ or both in $Y\cup\{c\}$. Then partition X into $A\cup B$ with $|A|=|B|=m$, such that no triangle of P with vertex b has both of its other vertices in A or both in B. Finally, use the vertex c to partition Y into $C\cup D$ in a similar way. Thus P may be expressed as $P_1\cup\cdots\cup P_{10}$, where:

P_1 consists of triangles with exactly two vertices in $X\cup\{a, b\}$;
P_2 consists of triangles with exactly two vertices in $Y\cup\{a, c\}$;
P_3 consists of triangles with two vertices in $A\cup\{b\}$ and one in B;

P_4 consists of triangles with two vertices in $B \cup \{b\}$ and one in A;
P_5 consists of triangles with two vertices in $C \cup \{c\}$ and one in D;
P_6 consists of triangles with two vertices in $D \cup \{c\}$ and one in C;
P_7–P_{10} consists of triangles all of whose vertices are in A, B, C, D respectively.

Consider the partial structure P_1. Regarding the vertices in $Y \cup \{c\}$ as colours, P_1 represents a proper partial edge-colouring of the complete graph on $X \cup \{a, b\}$, using at most $2m+1$ colours. By Theorem 10, P_1 is the union of at most six subsets, each of which can be completed to a proper edge-colouring of K_{2m+2}, i.e., to a partial STS on V involving every edge except those joining vertices in $Y \cup \{c\}$. But $2m+1$ is admissible, so we can complete each of these partial STS's to an STS on all $4m+3$ points.

Thus P_1 is the union of at most six extensible structures. A similar argument shows that the same is true of P_2.

Consider next P_3. Regarding the vertices in B as colours, this is a partial edge-colouring of K_{m+1} using at most m colours, which by [1, Corollary 4.3.3] can be extended to an edge-colouring of K_{2m+2} using $2m+1$ colours, and thence (as before) to an STS on all $4m+3$ points. Thus P_3 is extensible, and similarly P_4, P_5 and P_6 are extensible.

Finally, P_7, P_8, P_9 and P_{10} are extensible by [2, Theorem 1].

It follows that P is the union of at most twenty extensible structures. $\square$

8. Cayley tables

For any $n > 1$, let L be an $n \times n$ latin square on a set C of n symbols. It is a *free Cayley table* if the rows and columns can be labelled with the elements of C (not necessarily in the same order) such that the result is the Cayley table for a group structure on C. More precisely, we require the existence of bijections s, t from C to $\{1, \ldots, n\}$ such that C is a group under the binary operation $\circ$ given by

$$a \circ b = L(s(a), t(b)) \qquad (a, b \in C).$$

Thus the element in the ith row and jth column of L is $s^{-1}(i) \circ t^{-1}(j)$.

The *free Cayley table problem of order n*, $\mathscr{F}_n$, is the concentration of $\mathscr{L}_n$ obtained by restricting the goal structures to be the free Cayley tables on C. Thus the partial structures are the same as those of $\mathscr{L}_n$, namely the partial latin squares.

A free Cayley table is *cooperative* if the bijections s, t are equal; *well-positioned* if $s^{-1}(1)$ and $t^{-1}(1)$ are the identity element of the group structure; and *of given identity* if there is an element e of C, prescribed in advance, which is the identity element of the group structure. Any set of these conditions may be imposed on the goal structures to form a concentration of $\mathscr{F}_n$, and it is largely a matter of taste which set is regarded as the most 'natural' form of a Cayley table problem. We shall denote the presence of these conditions by attaching the following

superscripts:
c for 'cooperative',
w for 'well-positioned',
g for 'of given identity'.

If the condition of given identity is imposed in conjunction with one or both of the others, then an unfair problem is sometimes created, because the requirement of a group structure sometimes excludes e from being placed in certain cells and requires e to be placed in others. The restrictions which are required in order to obtain a fair problem are as follows.

Fair $\mathscr{F}_2^{cg}$: e not to be placed in the $(1,2)$ or $(2,1)$ cell; the other symbol not to be placed in the $(1,1)$ or $(2,2)$ cell.

Fair $\mathscr{F}_n^{wg}$: e not to be placed in the $(1,j)$ or $(j,1)$ cell if $j \neq 1$; no symbol other than e to be placed in the $(1,1)$ cell; if $n=2$, then in addition no symbol other then e may be placed in the $(2,2)$ cell.

Fair $\mathscr{F}_n^{cwg}$: if n is even, the same restrictions as Fair $\mathscr{F}_n^{wg}$; if n is odd, then in addition e may not be placed in the (j,j) cell if $j \neq 1$; furthermore, if $n=3$, then no symbol other than e may be placed in the $(2,3)$ or $(3,2)$ cell.

Since imposing successive conditions forms successive concentrations of the problem, Theorem 8 gives a set of inequalities between the problems $\mathscr{L}_n$, $\mathscr{F}_n$, $\mathscr{F}_n^c$, $\mathscr{F}_n^w$, $\mathscr{F}_n^g$, $\mathscr{F}_n^{cw}$ and (except when $n=2$) $\mathscr{F}_n^{cg}$; also, if n is even, we have $\kappa(\text{Fair } \mathscr{F}_n^{wg}) \leq \kappa(\text{Fair } \mathscr{F}_n^{cwg})$.

For each $n \geq 2$, the eight problems created by imposing none, any or all of the conditions c, w, g, then restricting the domain if necessary to obtain a fair problem, are called *the fair Cayley problems of order n*.

Theorem 18. (i) *The fair Cayley problems of order n have* $\kappa \leq \tau \leq n$.

(ii) *For each* $m \geq 1$, *the problems* $\mathscr{F}_{2m+1}^c$, $\mathscr{F}_{2m+1}^{cw}$ *and* $\mathscr{F}_{2m+1}^{cg}$ *are* $(2m+1)$-*regular.*

Proof. (i) Any completed or partially completed single row which is a partial structure is extensible in each case. Thus there is a template of cardinality n, and the result follows from Theorem 1(ii).

(ii) In any group of order $2m+1$, the equation $b^2 = c^2$ implies $b^{2m} = c^{2m}$ and hence $b = c$. Thus the partial structures in which the cells $(1,1), (2,2), \ldots, (2m+1, 2m+1)$ are each filled with the same symbol and all other cells are empty, is decadent for $\mathscr{F}_{2m+1}^c$, $\mathscr{F}_{2m+1}^{cw}$ and $\mathscr{F}_{2m+1}^{cg}$, and the result follows from Corollary 1.1. $\square$

The next theorem establishes precisely the intricacies of the fair Cayley problems of orders 2 and 3.

Theorem 19. (i) Fair $\mathscr{F}_2^{cg}$, Fair $\mathscr{F}_2^{wg}$ *and* Fair $\mathscr{F}_2^{cwg}$ *are simple problems, the other* *five fair Cayley problems of order* 2 *being of intricacy* 2;

(ii) $\kappa(\mathscr{L}_3)=\kappa(\mathscr{F}_3)=\kappa(\mathscr{F}_3^{\mathrm{w}})=\kappa(\mathscr{F}_3^{\mathrm{g}})=\kappa(\text{Fair } \mathscr{F}_3^{\mathrm{cwg}})=2$, *the other four fair Cayley problems of order* 3 *being of intricacy* 3.

Proof. (i) Fair $\mathscr{F}_2^{\mathrm{cg}}$, Fair $\mathscr{F}_2^{\mathrm{wg}}$ and Fair $\mathscr{F}_2^{\mathrm{cwg}}$ each have a unique goal structure, and are consequently simple. The other fair Cayley problems of order 2 have intricacy at most 2 by Theorem 18, and each have the partial latin square

$$\begin{bmatrix} c_1 & - \\ - & c_2 \end{bmatrix}$$

as a partial structure, which is clearly a failure.

(ii) To show that $\kappa(\mathscr{L}_3)=2$, we consider one failure from each equivalence class of failures under permutations of rows, columns and symbols, and check that it is the union of two extensible structures. This is straightforward, though a little tedious, and the details are not given.

Now, modulo permutations of the symbols c_1, c_2, c_3, the only two 3×3 latin squares are

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_1 \\ c_3 & c_1 & c_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{bmatrix}.$$

All the symbol-permutations of each of these are goal structures of $\mathscr{F}_3$, $\mathscr{F}_3^{\mathrm{w}}$ and $\mathscr{F}_3^{\mathrm{g}}$, and thus $\mathscr{L}_3=\mathscr{F}_3=\mathscr{F}_3^{\mathrm{w}}=\mathscr{F}_3^{\mathrm{g}}$.

In the case of Fair $\mathscr{F}_3^{\mathrm{cwg}}$, the restriction of the domain already noted implies that a template of order 2 may easily be constructed, and since Fair $\mathscr{F}_3^{\mathrm{cwg}}$ is not simple, its intricacy is 2.

The other four fair Cayley problems of order 3 have intricacy at most 3 by Theorem 18. The result follows from the fact that each of them has a decadent partial structure of cardinality 3, namely

$$\begin{bmatrix} c_1 & - & - \\ - & c_1 & - \\ - & - & c_1 \end{bmatrix}$$

in the case of $\mathscr{F}_3^{\mathrm{c}}$, $\mathscr{F}_3^{\mathrm{cw}}$ and $\mathscr{F}_3^{\mathrm{cg}}$, and

$$\begin{bmatrix} - & - & - \\ - & c_2 & c_3 \\ - & - & c_2 \end{bmatrix}$$

in the case of Fair $\mathscr{F}_3^{\mathrm{wg}}$ (where c_2, c_3 are the two non-identity symbols). □

Theorem 20. $\kappa(\text{Fair } \mathcal{F}_n^{\text{cwg}}) \leq n-1$ *for all* $n \geq 2$, *with equality when* n *is odd.*

Proof. Assume initially that $n \geq 5$. Let i and j be distinct integers between 2 and n inclusive, and consider a partial structure P_i of Fair $\mathcal{F}_n^{\text{cwg}}$ created by assigning symbols to some or all of the cells in row i, and (possibly) to the cell $(1, j)$. Since this is a partial structure of Fair $\mathcal{F}_n^{\text{cwg}}$, the $(i, 1)$ and $(1, j)$ entries (if they exist) are not the identity. Then P_i is non-extensible if the $(i, 1)$ and $(1, j)$ entries are filled with the same symbol, and is extensible otherwise. Moreover, we may also assign the identity symbol to the $(1, 1)$ entry without affecting extensibility.

Thus if P is any partial structure of Fair $\mathcal{F}_n^{\text{cwg}}$, it may be expressed as $\bigcup_{i=2}^n P_i$ where each P_i is extensible, provided that there exists a bijection $f: \{2, \ldots, n\} \to \{2, \ldots, n\}$ such that (for each $i = 2, \ldots, n$) $f(i) \neq i$ and, if the $(i, 1)$ and $(1, j)$ entries are filled with the same symbol in P, then $f(i) \neq j$.

Consider the bipartite graph with vertices labelled $2_A, 3_A, \ldots, n_A$, $2_B, 3_B, \ldots, n_B$, with i_A adjacent to j_B unless $i = j$ or the $(i, 1)$ and $(1, j)$ entries in P are filled with the same symbol. Let $A = \{2_A, \ldots, n_A\}$, $B = \{2_B, \ldots, n_B\}$, and for each subset S of A let $\varphi(S)$ be the set of all elements of B that are adjacent to some element of S. If $|S| = 1$ or 2 then $|\varphi(S)| \geq n-3$, while if $|S| \geq 3$, then $\varphi(S) = B$. Thus by Hall's Marriage Theorem [15, Chapter 8, Theorem 25A], there is a complete matching between A and B (since $n \geq 5$), and hence a bijection f obeying the required conditions. Thus $\kappa(\text{Fair } \mathcal{F}_n^{\text{cwg}}) \leq n-1$.

The cases $n = 2, 3$ are covered by Theorem 19. In the case $n = 4$, an exhaustive analysis (not given here) of those maximal partial structures for which there is no complete matching of the type described above, yields the result that $\kappa = 3$.

Now let $n = 2m+1$, and let a be any symbol other than the identity. If b, c are any elements of a group of order $2m+1$, and $b^2 = c^2$, then $b^{2m} = c^{2m}$ and hence $b = c$. Thus the partial structure in which each of the cells $(2, 2), (3, 3), \ldots, (n, n)$ has the symbol a and all other cells are empty, is decadent in Fair $\mathcal{F}_n^{\text{cwg}}$. Hence $\kappa(\text{Fair } \mathcal{F}_n^{\text{cwg}}) = n-1$ in this case. $\square$

Finally, a Cayley table is *pre-structured* if the bijections s, t are prescribed in advance and no other conditions are imposed. Since it does not matter which particular bijections are prescribed, we shall take them to be: $s(c_i) = t(c_i) = i$ $(i = 1, 2, \ldots, n)$. This Cayley table problem of order n is denoted by $\mathcal{F}_n^{\text{p}}$.

Theorem 21. $\mathcal{F}_n^{\text{p}}$ *is an* n*-regular problem for each* $n \geq 2$.

Proof. The identity is the only idempotent of a group, so that the partial structure

$$\begin{bmatrix} c_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c_n \end{bmatrix}$$

is decadent.

For each $k=1,\ldots,n$ let G_k be the latin square in which (for each $i, j=1,\ldots,n$) the (i,j)th cell contains the entry c_{i+j-k}, interpreting the subscript modulo n. Then each G_k is a Cayley table, with the given row and column symbols, for the cyclic group of order n, the identity in the case of G_k being c_k. This set of goal structures forms a template of order n, and the result follows from Corollary 1.1. □

9. Arcs and conics in finite projective planes

Let q be a power of an odd prime; the projective plane over GF(q) is denoted by PG($2, q$). For $k\geq 3$, a *k-arc* in GF(q) is a set of k points no three of which are collinear; a *conic* is the set of points defined by a quadratic form; an *oval* is an arc of maximum cardinality. By [9, Lemma 7.2.3 and Theorem 8.2.4], with q odd the conics and ovals are identical, and they are the $(q+1)$-arcs. Thus the problem of constructing conics in PG($2, q$) (which we denote by $\mathscr{C}(2, q)$) has PG($2, q$) as domain, the arcs as the partial structures and the conics as the goal structures. The failures are the complete (i.e., maximal) arcs of non-maximum cardinality. Thus it follows from [9, Lemma 9.4.1] that $\mathscr{C}(2, 3)$ and $\mathscr{C}(2, 5)$ are simple, and by [9, Lemma 9.4.3 and Theorem 9.44] that for every other odd prime power, $\mathscr{C}(2, q)$ is intricate.

Remark 3. Aut($\mathscr{C}(2, q)$) acts transitively on 4-element partial structures, so that for all odd prime powers q we can immediately deduce that $\varphi(\mathscr{C}, (2, q))\geq 4$; this multiple transitivity would seem to be a fairly general feature of construction problems involving finite geometries. In our case, we have the following stronger result if $q\neq 3$.

Lemma 7. (i) $\omega(\mathscr{C}(2, q))\leq\min(q, \lceil q-\frac{1}{4}\sqrt{q}+\frac{7}{4}\rceil)$ *whenever q is an odd prime power.*
(ii) *If in addition $q\neq 3$, then $\varphi(\mathscr{C}(2, q))\geq 5$.*

Proof. (i) This follows from [9, Theorem 10.4.4], together with the fact that the conics are exactly the $(q+1)$-arcs.
(ii) This follows from the Corollary to [9, Theorem 7.2.1]. □

Lemma 8. *For any odd prime power q, there exists a nonsingular cubic in* PG($2, q$) *with an even number of points (at least $q+1$) and possessing at least one inflexion.*

Proof. Denote the additive and multiplicative identities in GF(q) by $\mathbf{0}$, $\mathbf{1}$ to distinguish them from integers. Let f be the function from GF(q)$\backslash\mathbf{0}$ to GF(q) defined by

$$f(x)=x^{-1}(x+1)(x-1),$$

and for each x in GF(q) let $R(x)$ be the number of distinct square roots of x in GF(q) (which is 1 if $x=\mathbf{0}$ and is 0 or 2 otherwise). Now let c be a fixed element of GF(q) with $R(c)=0$. (We may, for example, choose c to be a generator of the multiplicative group GF(q)$\setminus\mathbf{0}$.) Then for all x in GF(q),

$$R(x)+R(cx)=2. \tag{9}$$

Now by [9, Theorem 11.7.1, parts (i), (ii)], the following cubics are nonsingular, with at least one inflexion each:

$$Q: x_2^2x_1+x_0^3-x_0x_1^2=0,$$
$$Q_c: x_2^2x_1+x_0^3-c^2x_0x_1^2=0.$$

The points in PG(2, q) represented by $(\mathbf{0}, \mathbf{1}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{0}, \mathbf{1})$ belong to both cubics, and to find the other points on Q, Q_c we may set $x_0=\mathbf{1}$ to obtain the equations

$$x_2^2=x_1^{-1}(x_1+1)(x_1-1),$$
$$x_2^2=x_1^{-1}(cx_1+1)(cx_1-1)$$

respectively. Thus Q has $2+\sum_{x\neq\mathbf{0}} R(f(x))$ points and Q_c has $2+\sum_{x\neq\mathbf{0}} R(cf(cx))=2+\sum_{y\neq\mathbf{0}} R(cf(y))$ points. There are two values of x at which $R(f(x))=1$ and two values of y at which $R(cf(y))=1$, so each cubic has an even number of points. Finally, by Eq. (9), they have an average of $q+1$ points each. □

Theorem 22. *Let $q\neq 3$ be an odd prime power. Then:*

$$\left\lceil\frac{q+1}{12}\right\rceil\leqslant\kappa(\mathscr{C}(2,q))\leqslant\min\left(\left\lceil\frac{q}{5}\right\rceil, \lceil\tfrac{1}{5}(q-\tfrac{1}{4}\sqrt{q}+\tfrac{7}{4})\rceil\right).$$

Proof. The upper bound follows from Theorem 1(iii) and Lemma 7. The lower bound arises from [14, Theorem 9.1], which states that if Q is a nonsingular cubic with a point of inflexion 0, then the points of Q form an abelian group structure determined by the property that $A+B+C=0$ if and only if A, B, C are collinear. (The fact that the context of [14] is that of an algebraically closed field does not affect this result.) By Lemma 8 we may choose such a cubic to have an even number of points at least equal to $q+1$; then there is a subgroup S of index 2 in Q, and $Q\setminus S$ forms an arc of order at least $(q+1)/2$.

Now by Bézout's Theorem (see, for example, [9, Section 10.1]), no conic can meet Q is more than 6 points. Thus at least $\lceil(q+1)/12\rceil$ conics are required to cover the arc $Q\setminus S$. □

This lower bound is due to P.M. Neumann.

10. Highly decadent problems

Clearly a fair uniform problem of value ν has decadence at most $\nu-1$. It is easy to generate artificial situations where this bound is achieved: for example,

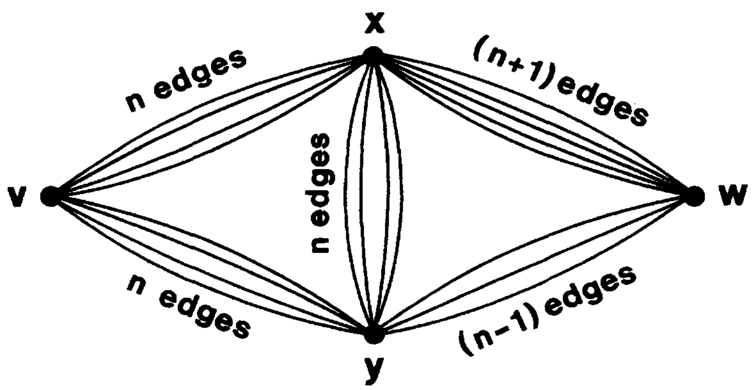

Fig. 10.

consider a problem whose domain is the set of cells of a $\nu \times (\nu - 1)$ matrix, whose partial structures are the sets of cells lying either in a single row or in a single column, and whose goal structures are the columns. However, if we demand that the partial structures are derived from a plausible method of trying to construct goal structures, it does not seem easy to achieve a decadence of $\nu - 1$ for arbitrarily high ν. It is fairly easy to find examples where $\delta \simeq \frac{1}{2}\nu$; Theorems 9 and 14 provide two such families. A further example is the family $\{\text{Pack}\,\mathscr{X}_n\}$ of maximum flow problems where the goal structures of $\mathscr{X}_n$ are the vw-paths in the graph of Fig. 10. (Here, the maximum number of disjoint vw-paths is $2n$, and any set of disjoint paths each of which uses an xy-edge is decadent.)

In the case of 1-factor problems (which, we recall, may be expressed in the form 'Pack $\mathscr{E}(G)$') we can do rather better than Theorem 9. For the graphs G_i $(i = 1, 2, 3)$ of Fig. 11 we have $\delta(\text{Pack}\,\mathscr{E}(G_i)) = i + 1$ and $\nu(\text{Pack}\,\mathscr{E}(G_i)) = i + 2$, the sets of horizontally drawn edges being decadent in each case, and Theorem 23 describes an infinite family of 1-factor problems where δ is bounded below by $\lfloor (2\nu + 1)/3 \rfloor$.

For each integer $n \geq 3$, construct the graph Z_n as follows:

If $n = 3m$, take a path of $4m - 1$ vertices and join its odd vertices to successive vertices of a $(2m + 1)$-circuit. (Fig. 12.)

If $n = 3m + 1$, treat a path of $4m + 1$ vertices and a $(2m + 1)$-circuit similarly. (Fig. 13.)

If $n = 3m + 2$, treat a path of $4m + 1$ vertices and a $(2m + 3)$-circuit similarly. (Fig. 14.)

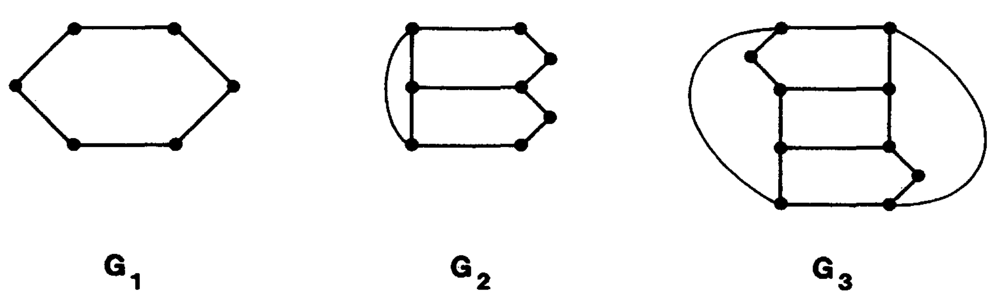

Fig. 11.

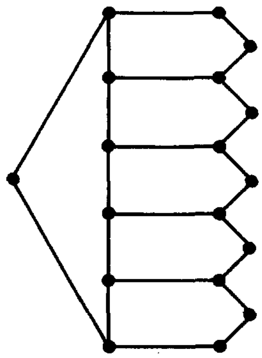

Fig. 12.

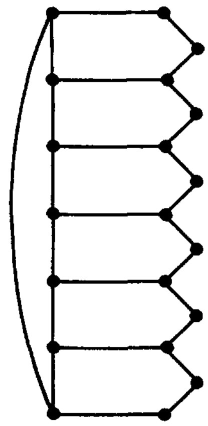

Fig. 13.

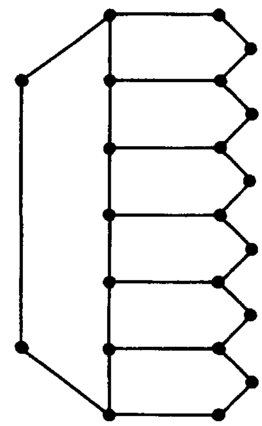

Fig. 14.

Theorem 23. *For* $n \geqslant 3$,

$$\nu(\text{Pack}\,\mathscr{C}(Z_n)) = n, \qquad \delta(\text{Pack}\,\mathscr{C}(Z_n)) \geqslant \lfloor (2n+1)/3 \rfloor.$$

Proof. In each case, any one of the edges joining the path to the circuit uniquely determines a 1-factor, and this 1-factor does not contain any other such edge. □

Note added in proof. Theorem 13 has now been improved to

$$\kappa(\text{Pack}^2\,\mathscr{C}(K_{2m})) \leqslant \frac{12m}{7}.$$

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