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Comments on the exact solutions of Mathieu's equation by D. J. Daniel (2020)

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So-called exact solutions of Mathieu's equation proposed by D. J. Daniel [Prog. Theor. Exp. Phys. **2020**, 043A01 (2020)] are flawed and not computable.

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1. Mathieu's equation

Mathieu's equation is a second-order differential equation of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (A - 2Q\cos 2x)y = 0\,, \tag{1}$$

with parameters A and Q. Its exact analytic solutions have not been developed. However, recently, Daniel [1] proposed two linearly independent closed-form analytic solutions.

Daniel's interest was a plane quantum pendulum; its wavefunction Ψ satisfies the one-dimensional time-independent Schrödinger equation of [1]

$$-\frac{\hbar^2}{2ml^2}\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + mgl(1-\cos\theta)\Psi = E\Psi\,; \tag{2}$$

see Ref. [1] for its notations. Then, with a change of variable $x = \theta/2$ and restructure of other variables, Eq. (2) can be transformed to Mathieu's equation.

With further change of variable $z = \exp(\mathrm{i}x)$ and by introducing an 'ansatz' of [1]

$$\Psi(z) = z^p \exp(-\alpha/z + \beta z) f(z)\,, \tag{3}$$

Daniel achieved an auxiliary function $f(z)$ of [1]

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + \left[\frac{2\alpha}{z^2} + \frac{\delta}{z} + 2\beta\right]\frac{\mathrm{d}f(z)}{\mathrm{d}z} + \left[\beta^2 - \frac{\omega^2}{2} + \frac{\beta\delta}{z} + \frac{\gamma}{z^2} + \frac{\alpha(\delta-2)}{z^3} + \frac{\alpha^2-\omega^2/2}{z^4}\right] \times f(z) = 0\,, \tag{4}$$

which formed the basis for his solution method. Many of the new variables in Eqs. (3) and (4) are arbitrary free parameters to be determined.

2. Daniel's first solution

Daniel's first solution selected $\alpha = \pm\omega/\sqrt{2}$, $\beta = \mp\omega/\sqrt{2}$, and $\delta = 2$, to simplify Eq. (4). Then, by the substitution of $z = r - su$ with arbitrary r and s, Eq. (4) of $f(z)$ was transformed to another second-order differential equation of $f(u)$ customised to his first solution, but that is not shown here for brevity. Further, he adopted a Laurent-series expansion for $f(u)$ [1]:

$$f(u) = \sum_{n=-\infty}^{\infty} \Gamma(n+\nu) a_n u^{-n-\nu}\,, \tag{5}$$

where $\Gamma(\,\cdot\,)$ is the gamma function; a_n is the expansion coefficient; and ν is arbitrary. Substitution of Eq. (5) to the differential equation of $f(u)$ resulted in a four-term recurrence relation for a_n, instead of the usual three-term recurrence relations of Mathieu's equation.

Daniel found that the recurrence relation for his a_n could be matched to one of the four-term recurrence relations for hypergeometric functions of the second order studied by Exton [2]. Daniel's interest was Exton's four-term recurrence relation for P_n [1,2]:

$$P_n(a, b; c; t) = \frac{t^n}{n!}\, {}_3F_1\left(a, b, -n; c; -\frac{1}{t}\right)\,, \tag{6}$$

where ${}_3F_1(\,\cdot\,)$ is a generalised hypergeometric function ${}_\rho F_q(\,\cdot\,)$ with $\rho = 3$ and $q = 1$. Note that Eq. (6) contains a factorial $n!$ and Eq. (5) has $\Gamma(n + \nu)$.

To match the two recurrence relations, Daniel made several selections of his free parameters such as a, b, c, r, s, γ, ν, and $t = r/s$. This process resulted in his $a_n \equiv P_n$ of Exton. He then inserted a_n to $f(u)$ in Eq. (5) and finally to the ansatz of $\Psi(z)$ in Eq. (3). The outcome is Daniel's first solution $\Psi_{\mathrm{I}}(x)$ with $z = \exp(\mathrm{i}x)$ [1]:

$$\Psi_{\mathrm{I}}(x) = z^{-1/2} \exp\left(-\frac{\alpha}{z} + \beta z\right) \sum_{n=-\infty}^{\infty} \left(\frac{t}{s}\right)^n {}_3F_1\left(a, b, -n; c; -\frac{1}{t}\right)(r - z)^{-n}\,, \tag{7}$$

which was proposed as an exact solution to both Mathieu's equation and Schrödinger's equation of quantum pendulum. For definitions of the parameters of Eq. (7), see Ref. [1].

Equation (7) requires the computation of ${}_3F_1(\,\cdot\,)$. However, for $\rho > q + 1$, the generalised hypergeometric function ${}_\rho F_q(\,\cdot\,)$ diverges for all non-zero arguments in general [3], except when its numerator parameters are non-positive integers. Furthermore, to make $a_n \equiv P_n$, one of Daniel's choices was $\nu = 1$, without realising that the gamma function is not defined at non-positive integers. Here, a paragraph of his is quoted verbatim [1]:

> "It is worth pausing here in order to affirm that the range of summation of n from $-\infty$ to ∞ in the Laurent series expansion of $f(u)$ is in no way affected by the $n!$ appearing in the denominator of the above result for a_n. Simply insert the value $\nu = 1$, as deduced from the first step above, into the gamma function $\Gamma(n + \nu)$ appearing in the Laurent series expansion of $f(u)$ to see that $\Gamma(n + \nu) = n!$, which clearly cancels the $n!$ contained in the denominator above for a_n."

His $\nu = 1$ is also instrumental in determining other free parameters; without it, Eq. (7) is not obtained. Anyway, his procedure involved a factorial of a negative integer. Daniel's first solution Ψ_{I} is therefore problematic.

3. Daniel's second solution

Daniel's second solution selected $\alpha = \pm\omega/\sqrt{2}$ to simplify Eq. (4), but β and δ were undecided at the beginning. Then, the following expansion of [1]

$$f(z) = \sum_{n=-\infty}^{\infty} (-1)^n \lambda^n a_n z^{1-n} \tag{8}$$

was assumed as a solution to Eq. (4). Here, λ is a free parameter and a_n is now the expansion coefficient of his second solution. Substitution of Eq. (8) to Eq. (4) led to another four-term recurrence relation for a_n.

Daniel again discovered that the recurrence relation for a_n of his second solution could be matched to another four-term recurrence relations for G_n by Exton [2]. Then, G_n of Exton is expressed in terms of a generalised hypergeometric function ${}_2F_2(\,\cdot\,)$ [1,2]:

$$G_n(a,b;c;t)=\frac{(a)_n(b)_n}{(c)_n n!}\,{}_2F_2(a+n,b+n;c+n,1+n;t)\,,$$

where $(\,\cdot\,)_n$ is the Pochhammer symbol. Note that both $(\,\cdot\,)_n$ and $n!$ are defined for $n\geq 0$.

To match the two recurrence relations, Daniel determined his free parameters such as a, b, c, t, β, δ, and λ. This procedure led to his second-solution $a_n\equiv G_n$ of Exton. The final outcome is Daniel's second solution of $\Psi_{\rm II}(x)$ with $z=\exp(\mathrm{i}x)$ [1]:

$$\Psi_{\rm II}(x)=z^{(1/2)+\delta}\exp\left(-\frac{\alpha}{z}+\beta z\right)\sum_{n=-\infty}^{\infty}(-\lambda)^n\frac{(a)_n(b)_n}{(c)_n n!}\,{}_2F_2(a+n,b+n;c+n,1+n;t)z^{-n}\,,$$

which was offered as another exact solution to Eqs. (1) and (2): for its parameters, see Ref. [1]. Subsequently, Daniel claimed that his two solutions $\Psi_{\rm I}$ and $\Psi_{\rm II}$ were independent.

Daniel's second solution contains both $n!$ and $(\,\cdot\,)_n$ evaluated for $n<0$, shortcomings attributed to Exton [2]; hence, it is not computable. For his first solution, Daniel was wary of $n!$ under $\sum_{n=-\infty}^{\infty}$; however, he did not make comments on this for his second solution.

4. Conclusion

Daniel's exact solutions [1] of Mathieu's equation involve the gamma function of non-positive integers, the factorial of negative integers, and Pochammer symbols with a negative number of factors; therefore they are flawed.

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