Healy, M. J. R. 1957. A rotation method for computing canonical correlations. *Mathematics of computation American Mathematical Society.* 11, pp. 83-86.

In one-dimensional problems of mathematical physics the word "adjacent" can here be usually interpreted positionally on the matrix itself, if the rows and columns are in an appropriate order. Otherwise the word should be interpreted in terms of the physical application. For example, in the inverse of the matrix L_n treated by W. L. Wilson [3], which arises from a two-dimensional physical problem, the elements at the positions (4, 14) and (7, 19) may be regarded as adjacent because they correspond to *pairs* of points that are close together in the physical problem. This question of the definition of "adjacent" is related to that of how to extrapolate from a matrix to a larger one. It seems difficult to formulate any general rule for extrapolation. For the present each case would have to be treated on its own merits by the exercise of judgment, as in the reference cited.

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1. I. J. GOOD, "Bounded integral transforms," *Quart. Jn. of Math.*, Oxford, v. 1, 1950, p. 185–190.
2. HAROLD HOTELLING, "Some new methods in matrix calculation," *Annals Math. Stat.*, v. 14, 1943, p. 1–34.
3. W. L. WILSON, JR., *Tables of Inverses to Laplacian Operators over Triangular Grids*, UMT FILE [*MTAC*, this issue, Rev. 53, p. 108].

A Rotation Method for Computing Canonical Correlations

Given two sets of variates, $x_i(i = 1, 2 \cdots p)$, $y_j(j = 1, 2 \cdots q)$ with $p \geqslant q$, Hotelling [1] has shown that it is possible to find linear transforms u_i, v_j of the x's and y's respectively with the properties

$$\begin{aligned} &1. \quad \text{var}\,(u_i) = \text{var}\,(v_j) = 1 \\ &2. \quad \text{cov}\,(u_i, u_k) = 0, \quad i \neq k \\ &\qquad \text{cov}\,(v_j, v_l) = 0, \quad j \neq l \\ &3. \quad \text{cov}\,(u_i, v_j) = 0, \quad i \neq j. \end{aligned}$$

The variates u and v are called canonical variates, and the correlations $\rho_i(i = 1, 2 \cdots q)$ between corresponding variates u_i, v_i are called canonical correlations.

The same problem may be stated in terms of matrix algebra. Suppose that the dispersion matrix of the x's and y's, considered as a single vector variate, is

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}' & \mathbf{B} \end{pmatrix}$$

where $\mathbf{A}$ has p rows and p columns, $\mathbf{B}$ has q rows and q columns, and $\mathbf{C}$ has p rows and q columns. The whole matrix is symmetric and positive definite. We seek matrices **U** of p rows and columns and **V** of q rows and columns such that

$$\mathbf{U\,A\,U'} = \mathbf{I} \quad \mathbf{V\,B\,V'} = \mathbf{I}$$
$$\mathbf{U\,C\,V'} = \mathbf{R}$$

where $R = (r_{ij}),\ r_{ii} = \rho_i,\ r_{ij} = 0,\ i \neq j$.

Hotelling showed that the ρ_i^2 were the roots of the equation

$$|\mathbf{C'\,A^{-1}\,C} - \lambda\mathbf{B}| = 0$$

and that the coefficients of $\mathbf{v}_i$ could be found as the solutions of the consistent homogeneous equations

$$(\mathbf{C'\,A^{-1}\,C} - \rho_i\,\mathbf{B})\mathbf{y} = 0;$$

the coefficients of u_i are then given by $\mathbf{A^{-1}\,C\,y}$. This procedure clearly presents a formidable computational problem if p and q are at all large. If a high-speed computer is available, the problem is greatly reduced, since the various stages form standard computational problems; $\mathbf{A^{-1}\,C}$ and $\mathbf{C'\,A^{-1}\,C}$ can be calculated by a process of pivotal condensation as described by A. C. Aitken [2], and the roots and vectors derived from $(\mathbf{C'\,A^{-1}\,C} - \lambda\mathbf{B})$ can be readily obtained from the latent roots and vectors of $\mathbf{B^{-1}\,C'\,A^{-1}\,C}$, or from those of the symmetric matrix $\mathbf{K^{-1}\,C'\,A^{-1}\,C\,K'^{-1}}$, where **K** is a triangular matrix such that $\mathbf{B} = \mathbf{K\,K'}$. Nevertheless, several fairly complicated stages of computation are involved, and their linking together is not altogether straightforward; in addition, a fair amount of intermediate storage capacity is called for. The purpose of this note is to describe a more direct method of computation which avoids some of these difficulties.

The method suggested here is similar to a well known rotation method for finding the latent roots and vectors of as ymmetric matrix (see Householder [3]). We wish to find matrices **U** of p rows and columns and **V** of q rows and columns such that

$$\begin{pmatrix} \mathbf{U} & 0 \\ 0 & \mathbf{V} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C'} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{U'} & 0 \\ 0 & \mathbf{V'} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{R'} & \mathbf{I} \end{pmatrix}.$$

Triangular matrices **H** and **K** can be found such that $\mathbf{A} = \mathbf{H\,H'}$ and $\mathbf{B} = \mathbf{K\,K'}$ (computationally, this is a straightforward and accurate process); writing $\mathbf{U\,H} = \mathbf{S}$ and $\mathbf{V\,K} = \mathbf{T}$, we obtain

$$\begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{T} \end{pmatrix} \begin{pmatrix} \mathbf{H}^{-1} & 0 \\ 0 & \mathbf{K}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C'} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \mathbf{H}^{-1} & 0 \\ 0 & \mathbf{K}'^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{S'} & 0 \\ 0 & \mathbf{T'} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{R'} & \mathbf{I} \end{pmatrix}$$

or

$$\begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{T} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{H^{-1}\,C\,K'^{-1}} \\ \mathbf{K^{-1}\,C'\,H'^{-1}} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{S'} & 0 \\ 0 & \mathbf{T'} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{R} \\ \mathbf{R'} & \mathbf{I} \end{pmatrix}.$$

In this form it is seen that we require two orthogonal matrices **S** and **T** such that they together transform the matrix $\mathbf{H^{-1}\,C\,K'^{-1}}$ of p rows and q columns to the diagonal form **R**. We can now show that **S** and **T** can be found iteratively as the products of sequences of elementary rotation matrices.

Writing for brevity $\mathbf{H^{-1}\,C\,K'^{-1}} = \mathbf{D}$, consider the matrix

$$\mathbf{D}_1 = \mathbf{S}_1\,\mathbf{D}\,\mathbf{T}_1',$$

where

$$\mathbf{S}_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \cdots 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ & \cdots & \\ 0 & 0 & 0 \cdots 1 \end{bmatrix}$$

$$\mathbf{T}_1 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \cdots 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ & \cdots & \\ 0 & 0 & 0 \cdots 1 \end{bmatrix}.$$

It is found that

$$\mathbf{D}_1 = \begin{bmatrix} d_{11}c_1c_2+d_{21}s_1c_2+d_{12}c_1s_2+d_{22}s_1s_2 & -d_{11}c_1s_2-d_{21}s_1s_2+d_{12}c_1c_2+d_{22}s_1c_2 & d_{13}c_1+d_{23}s_1\ldots \\ -d_{11}s_1c_2+d_{21}c_1c_2-d_{12}s_1s_2+d_{22}c_1s_2 & d_{11}s_1s_2-d_{21}c_1s_2-d_{12}s_1c_2+d_{22}c_1c_2 & -d_{13}c_1+d_{23}s_1\ldots \\ d_{31}c_2+d_{32}s_2 & -d_{31}s_2+d_{32}c_2 & d_{33} \\ & & \text{etc.,} \end{bmatrix}$$

where s_1, c_1, s_2, c_2 are written for $\sin\theta_1$, $\cos\theta_1$, $\sin\theta_2$, $\cos\theta_2$, only elements in the first two rows and columns being affected.

We now choose θ_1 and θ_2 to satisfy the equations

$$-d_{11}c_1s_2 - d_{21}s_1s_2 + d_{12}c_1c_2 + d_{22}s_1c_2 = 0$$
$$-d_{11}s_1c_2 + d_{21}c_1c_2 - d_{12}s_1s_2 + d_{22}c_1s_2 = 0.$$

With these values of θ_1 and θ_2, the transformation causes two of the elements of $\mathbf{D}_1$ to vanish.

A similar pair of transformations can now be applied to $\mathbf{D}_1$ to remove two other elements. This will in general restore to non-zero values the elements just dealt with; but it will be seen that each pair of transformations dealing with elements d_{ij} and d_{ji} leaves unaltered the sum of squares of the remaining non-diagonal elements. It can be shown that, on successive iteration, the sum of squares of all the non-diagonal elements will tend to zero and the matrix itself to diagonal form. The required orthogonal matrices are the products of all the successive rotation matrices.

The above argument holds good when $p = q$, so that $\mathbf{D}$ is a square matrix. If $p > q$, there will be x variates that have no corresponding y variates. These can be dealt with by a transformation of the type $\mathbf{D}_1 = \mathbf{S}_1\,\mathbf{D}$ which annihilates a single element and retains the sum of squares of the remaining non-diagonal elements as before.

The angles θ_1 and θ_2 are obtained from the equations given above. These are found to give

$$\tan 2\theta_1 = \frac{2(d_{11}d_{21} + d_{22}d_{12})}{d^2_{11} - d^2_{21} + d^2_{12} - d^2_{22}}$$

$$\tan 2\theta_2 = \frac{2(d_{11}d_{12} + d_{22}d_{21})}{d^2_{11} + d^2_{21} - d^2_{12} - d^2_{22}}$$

from which the sines and cosines can be derived without difficulty.

A point of practical importance is that the d's (which are, in fact, correlation coefficients) do not exceed 1 in magnitude; the same is true of the r's. For this reason, the iterative scheme can easily be programmed using fixed-point arithmetic, though double-precision working is frequently needed in the calculation of the sines and cosines.

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1. H. Hotelling, "Relations between two sets of variates," *Biometrika*, v. 28, 1936, p. 321–377;
2. A. C. Aitken, "The evaluation, with applications, of a certain triple product matrix,' Roy. Soc. Edinburgh, *Proc.*, v. 57, 1936, p. 172–181.
3. A. S. Householder, *Principles of Numerical Analysis* (Sect. 4.115), McGraw-Hill Book Co., Inc., New York, 1953.

Polynomial Approximations to Bessel Functions of Order Zero and One and to Related Functions

In the course of preparing a programme for a large digital computer, the following formulas, which are of use in the calculation of Bessel functions on such machines, have been obtained. The methods of obtaining the approximations are those described by Lanczos in NBS AMS 9 [1], with trivial modifications (a complete set of references is given in [1]). The results enable $J_0(x)$, $J_1(x)$, $Y_0(x)$, $xY_1(x)$, and $\bar{J}_0(x)$ to be calculated for all values of x and $K_0(x)$, $K_1(x)$, and $\bar{K}_0(x)$ to be calculated for values of x exceeding unity; where we define

$$\bar{J}_0(x) = \int_0^x J_0(t)dt$$

and

$$\bar{K}_0(x) = \int_0^x K_0(t)dt.$$

We define auxiliary functions such that

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left(P_n(x)\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) - Q_n(x)\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right), \qquad n = 0, 1,$$

$$Y_n(x) = \sqrt{\frac{2}{\pi x}}\left(P_n(x)\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + Q_n(x)\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right), \qquad n = 0, 1,$$

$$\bar{y}_n(x) = \left(J_n(x)\log\frac{x}{2} - \frac{\pi}{2}Y_n(x)\right), \qquad n = 0, 1,$$

$$K_n(x) = e^{-x}\sqrt{\frac{\pi}{2x}}G_n(x), \qquad n = 0, 1,$$

$$\bar{J}_0(x) = 1 - \sqrt{\frac{\pi}{2x}}\left(\bar{P}_0(x)\cos\left(x + \frac{\pi}{4}\right) - \bar{Q}_0(x)\sin\left(x + \frac{\pi}{4}\right)\right),$$

$$\bar{K}_0(x) = \frac{\pi}{2} - e^{-x}\sqrt{\frac{\pi}{2x}}\bar{G}_0(x).$$