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A Note on the Construction of Row-and-column Designs with Two Replicates

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SUMMARY

This paper shows that useful designs for variety trials with two replicates, each arranged in a compact array of plots, can be derived from existing row-and-column designs for two non-interacting sets of treatments. Combinatorial and statistical properties are preserved; in particular, if the original design is efficient, then so is the new design.

Keywords: BALANCE; CONTRACTION; EFFICIENCY FACTOR; NESTED DESIGN; OPTIMALITY; ORTHOGONALITY; ROW-AND-COLUMN DESIGN

1. INTRODUCTION

In this paper we draw attention to a little-known combinatorial relationship between two types of row-and-column design that have both been used in agricultural field experiments, although in very different circumstances.

One of the types of design was developed to allow the plants of a perennial crop to be used in two experiments on different occasions. An example, attributed by Preece (1971) to Dr G. H. Freeman, is in Table 1.

This design provides a single replication of two sets of treatments. One set, with six levels, is orthogonal to columns and balanced with respect to rows, while the other, with seven levels, is orthogonal to rows and balanced with respect to columns. The 42 plots are arranged in a 6×7 rectangle in the field. The two-way elimination of heterogeneity and the balance properties ensure that the effects of both sets of treatments can be estimated efficiently, provided that there is no interaction between them.

An example of the second type of design is in Table 2. The purpose of the experiment is to compare 42 varieties of an agricultural crop under field conditions. As often happens in the early stages of a plant breeding programme, seed and other resources are in short supply and the experimenters are restricted to two small plots of each variety. Each replication is accommodated on a compact block consisting of a 6×7 array of plots: the use of a row-and-column arrangement within each block allows fertility variations to be eliminated in two directions. Designs of this type are sometimes referred to as ‘nested’ row-and-column designs.

The statistical literature contains many examples of row-and-column designs for two sets of treatments (Clarke, 1963, 1967; Preece, 1966, 1971, 1976, 1982; Hedayat *et al.*, 1970).

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By contrast, there are few examples of nested row-and-column designs in two replications. A short catalogue of 20 such designs for up to 100 varieties has recently been published by Patterson and Robinson (1989), but without details of construction. 11 of the designs are known to be optimal in the sense that their efficiency factors are as large as possible; the others achieve a high standard of efficiency but there may still be room for improvement.

The purpose of the present paper is to show that every nested row-and-column design with two replications is combinatorially equivalent to a single-replicate row-and-column design for two sets of treatments. Using this equivalence simplifies the identification and construction of optimal nested designs. In particular, the best designs catalogued by Patterson and Robinson (1989) can be derived from known balanced designs for two sets of treatments, including general series described by Clarke (1963) and Hedayat *et al.* (1970). We also add a 12th optimal nested design to the catalogue.

TABLE 1
Design for two sets of treatments, one at six levels, the other at seven levels†

Row	Column (I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
(i)	1 Cα	2 Gβ	3 Eγ	4 Fδ	5 Aε	6 Dζ	7 Bγ
(ii)	8 Eβ	9 Dγ	10 Gζ	11 Aα	12 Cδ	13 Bε	14 Fβ
(iii)	15 Fγ	16 Cζ	17 Aβ	18 Gε	19 Bα	20 Eδ	21 Dδ
(iv)	22 Gδ	23 Eε	24 Dα	25 Bβ	26 Fζ	27 Aγ	28 Cε
(v)	29 Dε	30 Fα	31 Bδ	32 Eζ	33 Gγ	34 Cβ	35 Aζ
(vi)	36 Bζ	37 Aδ	38 Fε	39 Cγ	40 Dβ	41 Gα	42 Eα

†From Preece (1971), p. 427.

TABLE 2
Design for 42 varieties in two blocks of 6 × 7 plots

	Column (I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
Block 1 Row							
(i)	1	2	3	4	5	6	7
(ii)	8	9	10	11	12	13	14
(iii)	15	16	17	18	19	20	21
(iv)	22	23	24	25	26	27	28
(v)	29	30	31	32	33	34	35
(vi)	36	37	38	39	40	41	42
	Column A	B	C	D	E	F	G
Block 2 Row							
α	11	19	1	24	42	30	41
β	17	25	34	40	8	14	2
γ	27	7	39	9	3	15	33
δ	37	31	12	21	20	4	22
ε	5	13	28	29	23	38	18
ζ	35	36	16	6	32	26	10

2. RELATIONSHIP BETWEEN DESIGNS D_1 AND D_2

We refer to the design in Table 1 as D_1 and the design in Table 2 as D_2.

Design D_1 is completely identified by 42 quintuples, each consisting of a Roman number in lower case, a Roman number in upper case, a Latin letter, a Greek letter and an Arabic number. The Roman numbers provide the row and column labels, the Latin and Greek letters the labels of the treatments of the two sets and the Arabic numbers the plot labels.

These quintuples are also used in D_2, but the elements have different parts to play. Thus, the Roman numbers represent the rows and columns of block 1, the Greek letters provide the labels for the rows of block 2 and the Latin letters the column labels of block 2; the Arabic numbers now represent the 42 varieties.

For example, plot 34 of D_1 is in row (v), column (VI), and receives treatment C in one experiment, treatment β in the other. In D_2, variety 34 appears on the plot given by the intersection of row (v) and column (VI) in block 1 and on the plot given by the intersection of row β and column C in block 2.

3. ORTHOGONALITY AND BALANCE IN THE TWO DESIGNS

Design D_1 has the following properties of orthogonality and balance.

(a) Columns are orthogonal to rows.
(b) Latin letters (one set of treatments) are orthogonal to rows.
(c) Latin letters are balanced with respect to columns.
(d) Greek letters (the other set of treatments) are balanced with respect to rows.
(e) Greek letters are orthogonal to columns.
(f) Greek letters are orthogonal to Latin letters.

In translation, properties (a) and (f) simply mean that the two blocks of D_2 each contain a full complement of 42 plots, property (b) that each row in replicate 1 has exactly one variety in common with each column of replicate 2 and property (e) that each column of replicate 1 has exactly one variety in common with each row of replicate 2. Property (c) means that the column component of D_2, i.e. the block design obtained by ignoring rows (Pearce (1983), section 6.6) is optimal in that its efficiency factor is as large as possible for a design of this size. The similarly defined row component, a design for six treatments in blocks of seven, is also optimal; this is a consequence of property (d).

Taken together, properties (b) and (e) ensure that no varietal comparison in D_2 is confounded with both rows and columns. This type of orthogonality is called 'adjusted orthogonality' by John and Eccleston (1986) and John (1987) and simply 'orthogonality' by Patterson and Robinson (1989).

An orthogonal nested row-and-column design with optimal row component and optimal column component is itself optimal. Hence design D_2 as a whole is optimal; its efficiency factor is 0.6396, compared with 0.6358 for the design for 42 varieties catalogued by Patterson and Robinson (1989).

The relationship between design D_2 and design D_1 is a more complex version of the relationship that is known to exist between a two-replicate resolvable block design for ks varieties in blocks of k and its contraction, a symmetrical block design for s varieties in blocks of k. This relationship was used by Williams *et al.* (1976, 1977) to simplify the construction and identification of optimal two-replicate incomplete block designs. Our objectives in the present paper are similar. Optimal row-and-column designs for two sets of treatments are easier to recognize than optimal nested designs.

We use the term contraction both in the original sense and also to describe the relationship of D_1 as a whole to D_2. Thus, D_1 is the contraction of D_2. It is also a combination of two contractions. One, given by the Greek letters and rows, is the contraction of the row component of D_2; we call it the row contraction for short. The other, consisting of the Latin letters and columns, is the column contraction of D_2.

4. NESTED ROW-AND-COLUMN DESIGNS WITH BALANCED CONTRACTIONS

More generally, a two-replicate nested row-and-column design has two blocks, each consisting of an $m \times n$ array of plots, with one plot in each block for each of the $m \times n$ varieties. Without loss of generality we take $m \leqslant n$.

Several general series of contractions giving optimal nested designs are available. In the simplest, for any m not equal to 2 or 6 and $n = m$, the contractions are Graeco-Latin squares. These give the designs listed by Patterson and Robinson (1989) for 25, 49, 64, 81 and 100 varieties.

The construction described by Clarke (1963) gives a suitable contraction for $n = m + 1$ and m taking any value such that there is a set of three mutually orthogonal $m \times m$ Latin squares. This method gives the catalogued designs for 30, 56, 72 and 90 varieties. A more powerful construction described by Hedayat *et al.* (1970), also for $n = m + 1$, requires only two orthogonal $m \times m$ Latin squares with a common transversal and so is suitable for any m other than 2, 3 or 6. This yields potentially useful additional nested designs for 110 varieties in two 10×11 arrays of plots and 210 varieties in 14×15 arrays. Neither of these two methods includes the design for the 42 varieties given in Table 2.

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