Theory of Cyclic Rotation Experiments

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Summary

The paper reviews some of the statistical problems arising in the design and analysis of long-term cyclic experiments comparing different crop rotations. Three types of design are distinguished and their properties considered. These are basic designs with all phases of the rotations in each block, reduced designs with mutually exclusive groups of phases kept in separate blocks and phase-confounded designs in which some contrasts between test crops are partially confounded with block differences.

Methods of analysing the yields of test crops by estimating the mean effects of the rotations over the years and regressions on seasonal and time variates are discussed theoretically for replicated experiments of basic or reduced design. The analysis is complicated by correlations between yield values recurring on the same plots and by lack of homogeneity in residual year-to-year variations in rotation effects. The main topics considered are (1) the estimation of errors, (2) the losses of information due to using unweighted means and regressions ignoring the correlations, (3) methods for recovering this information.

1. Design of Rotation Experiments

1.1. *Introduction*

A rotation of crops is a system in which different crops are grown in sequence on the same land. A suitably chosen rotation gives a degree of natural control over certain factors, such as weeds, pests and diseases and general deterioration of soil condition, whose effects can accumulate under continuous cropping with the same or similar crops to such an extent that virtual crop failure results. Thus, for example, continuous cropping tends to restrict cultivations to a few types and times, thereby encouraging the spread of particular weeds.

Crop rotation has been practised in this country for more than 1,500 years. For much of that time rotations were enforced rigidly, often by law or contract, but the development in the twentieth century of other methods of maintaining fertility and controlling weeds, pests and diseases has widened the choice of rotations and created a growing need for rotation experiments.

Rotation experiments can be divided into two classes: (a) experiments comparing the effects of treatments on the crops of a single rotation and (b) experiments in which different rotations are compared.

The present paper is concerned primarily with statistical points arising in experiments of class (b). Until recently only a few experiments of this class had been attempted and there was little discussion of them in the literature. Crowther and Cochran (1942) described two experiments being done in the Sudan Gezira, probably the first experiments of the class with modern designs, and discussed their analysis. Yates (1952) discussed the analysis and results of an early experiment in Brazil, and Stevens (1956) considered the design and proposed analysis for later experiments of improved design. In two further papers Yates (1949, 1954) provided more general discussions on problems of design and analysis respectively.

Practical experience of the simpler class (a) experiments is much greater. Some points arising in the analysis of these experiments are also relevant to the analysis of class (b) experiments; these are considered by Cochran (1939) in a valuable general account of long-term agricultural experiments, and by Patterson (1953).

The terminology used by Yates (1954) will be followed throughout the present paper. In addition the term *rotation experiment* will be reserved for an experiment of class (b). An experiment of class (a) will be referred to as a *fixed-rotation experiment.*

1.2. *Short-term Rotation Experiments*

Rotation experiments need not be complex long-term experiments. Much valuable information can be obtained from relatively simple experiments with different sequences of crops grown on adjacent plots for a few years followed by one or more crops common to all plots.

A successful experiment of this type, primarily designed to test the effects of previous cropping on incidence of eyespot in winter wheat, is described by Glynne and Slope (1959). Table 1 shows the cropping scheme for one of the four replicate blocks of the experiment.

TABLE 1

One block of a short-term rotation experiment on eyespot

Year		Plot 1	2	3	4
1	Treatment crops	W	P	W	Be
2		W	W	P	P
3	Test crops	W_1	W_2	W_3	W_4

W: winter wheat. P: potatoes. Be: beans.

The crops grown in years 1 and 2 act as treatments and are called *treatment crops.* The four sequences of treatment crops give a 2×2 factorial scheme; (wheat in year 1 *v.* no wheat in year 1) $\times$ (wheat in year 2 *v.* no wheat in year 2).

The single crop grown in year 3 tests the effects of the four sequences and is therefore known as a *test crop.* A feature of the experiment, which is found in many rotation experiments, is that some of the treatment crops themselves act as *partial test crops*, providing information of a different type to the main test crop. Thus the winter wheat in year 2 compares the effects of wheat and potatoes grown in year 1.

Experiments of this type present few problems of design and analysis that are not covered by the existing theory of annual experiments. On the other hand, the more elaborate cyclic experiments discussed below present a number of problems with no analogy in annual experiments; these problems are the main concern of the present paper.

1.3. *Cyclic Rotation Experiments*

In experiments of this type the rotations of crops are grown in definite cycles of c years, where c may vary between rotations. Only rotations with one crop per year will be considered explicitly, but the extension to cropping cycles of shorter duration and to perennial crops does not introduce any new principle. Continuous cropping is treated as a rotation with $c = 1$.

Different rotations can be compared in any year in which they are growing the same crop. The test crops may be crops of commercial interest or they may be chosen for their sensitivity to some particular soil conditions that the rotations are expected to influence. Several different test crops can be included in a rotation to test different types of effect or the same test crop can be included more than once per cycle. Test crops need not be common to all the rotations of an experiment. For example, the rotations (1) continuous cotton, (2) alternate cotton and maize and (3) continuous maize can be included in the same experiment even though rotations (1) and (3) have no crop in common and cannot be compared directly. Cotton is the test crop for comparing rotations (1) and (2) and maize is the test crop for rotations (2) and (3). There is, however, clearly no point in including in the same experiment groups of rotations having no crop in common.

For any rotation there are c possible crop *sequences*, with the same cyclic order of cropping but at different points of the rotation in any particular year. These sequences are said to be in different *phase*. The term *phase* is also used to denote a particular component of the cropping cycle (Yates, 1954).

A fundamental principle in the design of a cyclic rotation experiment is that all sequences appertaining to each rotation should be included in the experiment, i.e. all phases should occur in each year. This principle is emphasized in all the papers mentioned in Section 1.1. In particular, Yates (1952) illustrated the difficulties of analysis and interpretation of a rotation experiment with incomplete phasing.

1.3.1. *Cyclic experiments for rotations of equal length*

The simplest type of cyclic rotation experiment compares the effects of varying some of the crops of a basic crop rotation. The crops that are common to all the rotations are test crops; the crops that are varied are treatment crops. Each sequence of the basic rotation is allocated to a *series* of plots, consisting of one or more replicate blocks, the different variants being allotted at random to the plots within the blocks. The blocks of a series need not be adjacent. Blocks in different series must be distinguished from replicate blocks within a single series.

Table 2 shows the cropping scheme for an experiment of this type being done, with minor variations between centres, at Rothamsted and on N.A.A.S. Experimental Husbandry Farms. The purpose of the experiment is to evaluate the effects on soil fertility of three different 3-year leys in comparison with a 3-year sequence of arable crops. The effects are compared by three test crops, wheat, potatoes and barley, grown in consecutive years after the third treatment year. The basic cropping cycle has a period of 6 years, so that six series of plots are required. Each series of plots consists of two replicate blocks.

This design gives within-block comparisons between the yields of the test crops, potatoes, wheat and barley, in each year except 3 preliminary years when the treatment crops are first grown.

Table 3 shows the rotations for an experiment of similar type and design at Woburn (an experimental farm run by Rothamsted). This brings out some additional points. Six different *tests* on the potatoes can be distinguished. Four of these tests, represented by P_1, P_2, P_3 and P_4, are of the same type as those of Table 2; the additional tests are represented by P'_3 and P'_4. Comparison of the yields of P'_3 with the yields of P'_4 gives information on the effects of the treatment crops in the third

TABLE 2

Cropping scheme for a long-term rotation experiment on the effects of leys on fertility

Series I	II	III	IV	V	VI	*Rotations* A	B	C	D	
Years										
1, 7, ...	2, 8, ...	3, 9, ...	4, 10, ...	5, 11, ...	6, 12, ...	Lg	Lc	Lu	H	Treat-
2, 8, ...	3, 9, ...	4, 10, ...	5, 11, ...	6, 12, ...	7, 13, ...	Lg	Lc	Lu	S.B	ment
3, 9, ...	4, 10, ...	5, 11, ...	6, 12, ...	7, 13, ...	8, 14, ...	Lg	Lc	Lu	O	crops
4, 10, ...	5, 11, ...	6, 12, ...	(1), 7, ...	(2), 8, ...	(3), 9, ...	W_1	W_2	W_3	W_4	Test
5, 11, ...	6, 12, ...	7, 13, ...	(2), 8, ...	(3), 9, ...	(4), 10, ...	P_1	P_2	P_3	P_4	crops
6, 12, ...	7, 13, ...	8, 14, ...	(3), 9, ...	(4), 10, ...	(5), 11, ...	B_1	B_2	B_3	B_4	

Lg: grazed ley. H: one-year grass for hay. P: potatoes.
Lc: conserved ley. S.B: sugar beet. W: wheat.
Lu: lucerne ley. O: oats. B: barley.

TABLE 3

Rotations for a ley–arable experiment

A	B	C	D
L′	Lu′	P'_3	P'_4
L″	Lu″	W'_3	W'_4
L‴	Lu‴	H	K
P_1	P_2	P_3	P_4
B_1	B_2	B_3	B_4

A and B: Ley rotations. C and D: Arable rotations.
L: ley. H: hay.
Lu: lucerne. K: kale.
P: potatoes. B: barley.
W: wheat.

There are eight plots per block, two for each rotation. On half of these the same rotation is repeated in each cycle; on the other half the ley and arable rotations are alternated (for details, see Cochran, 1939).

year. As P'_3 and P'_4 are in the same phase of the basic cropping cycle accurate comparisons can be made between them. Similarly W'_3 and W'_4 give information on fourth-year effects.

Comparisons between the yields of a crop in different phases of the basic rotation are confounded with blocks in any one year. Mean comparisons of this type over a complete cycle of 5 years do not involve constant block differences but must still be regarded as confounded because block effects vary considerably from year to year. For example, the mean differences (i) P_3' *v.* P_3, (ii) L′ *v.* L″ *v.* L‴ and (iii) P_1 *v.* P_3' are confounded with blocks × years. Differences such as (i) and (ii), between different phases of the same rotation, have been called *phase differences* by Yates (1954).

Other types of design have to be used (a) when the rotations differ in length and (b) to obtain full information on phase differences and other comparisons that are confounded in designs of the above type. Some of the possibilities are discussed in Sections 1.3.2 to 1.3.4. In case (b) two or more sequences must be included in each block for some or all of the rotations. The additional information so obtained may well be considered of insufficient value to compensate for the loss of accuracy on main test crop comparisons due to non-uniform cropping of the blocks in the test phases and increase in block size.

1.3.2. *Basic designs*

A *basic design* includes all the sequences of the rotations in a single block, i.e. there is no confounding. Basic designs are available for any set of rotations.

The rice–pasture experiment discussed by Yates (1954) provides an example of a basic design. The experiment compares three rotations:

A. 1 year rice, 2 years grass,

B. 2 years rice, 2 years grass,

C. 1 year rice, 3 years grass.

Table 4 shows the cropping scheme for one of three blocks of the experiment. As there are altogether 11 phases in the three rotations, each block consists of 11 plots.

TABLE 4

One replicate of a rice–pasture rotation experiment

	Rotations										
	A			B				C			
Years	1	2	3	4	5	6	7	8	9	10	11
1, 13, ...	R_1	G	G	R_2	G	G	R_3	R_4	G	G	G
2, 14, ...	G	R_1	G	R_3	R_2	G	G	G	R_4	G	G
3, 15, ...	G	G	R_1	G	R_3	R_2	G	G	G	R_4	G
4, 16, ...	R_1	G	G	G	G	R_3	R_2	G	G	G	R_4
5, 17, ...	G	R_1	G	R_2	G	G	R_3	R_4	G	G	G
6, 18, ...	G	G	R_1	R_3	R_2	G	G	G	R_4	G	G
7, 19, ...	R_1	G	G	G	R_3	R_2	G	G	G	R_4	G
8, 20, ...	G	R_1	G	G	G	R_3	R_2	G	G	G	R_4
9, 21, ...	G	G	R_1	R_2	G	G	R_3	R_4	G	G	G
10, 22, ...	R_1	G	G	R_3	R_2	G	G	G	R_4	G	G
11, 23, ...	G	R_1	G	G	R_3	R_2	G	G	G	R_4	G
12, 24, ...	G	G	R_1	G	G	R_3	R_2	G	G	G	R_4

R: rice (test crop). G: grass.

The different crops of rice that may be compared in this experiment are distinguished by suffixes in Table 4, including separate suffixes for the repeated test crop of rotation B. There are therefore four tests of the rice test crops to be compared. All comparisons R_1 *v.* R_2 *v.* R_3 *v.* R_4, including the phase difference R_2 *v.* R_3, can be made within blocks in every year after the fourth. All the yields of grass occurring in different phases of the same rotation or in different rotations can also be compared, if necessary. The whole experiment follows a cycle of period 12 years given by the lowest common multiple of the periods of the individual rotations.

1.3.3. *Reduced designs*

The main disadvantage of the basic design is the large size of block sometimes required. Under certain conditions this difficulty can be overcome by dividing the crop sequences into groups depending on their comparability. Two sequences, for the same or different rotations, will be described as *comparable*, with respect to a particular test crop, if there is at least 1 year with the test crop grown in both sequences. For example, sequence 1 of Table 4 is comparable with any sequence of rotations B and C, with respect to the rice test crop, but not with sequences 2 and 3. Sequence 4 is comparable with sequences 5, 7, 8 and 9 but not with sequences 6, 10 and 11. It should be noted, in considering rotations with a repeated test crop, that comparable sequences may or may not provide information on all possible types of difference. The terms *fully comparable* and *partially comparable* can be used to distinguish the two cases. For example, sequences 1 and 4 are fully comparable for the rice crop; sequences 4 and 8 are partially comparable.

Consider now the following procedure:

(1) Select one or more test crops whose contrasts are not to be confounded.
(2) Divide the full set of sequences for all the rotations into as many groups as possible such that any two sequences in different groups are not comparable with respect to any of the test crops selected in stage (1).
(3) Allocate each group of sequences to a different series of plots consisting of one or more replicate blocks.

A design constructed in this way with two or more groups of sequences will be called a *reduced design*. Some, all or none of the contrasts between a test crop not selected in stage (1) may be confounded automatically, depending on its position in the rotations relative to the selected test crops.

The sequences allocated to any series consist of a fraction $1/s$ of the sequences for each rotation, where s, the number of series, is a common factor of the periods of the rotations. If the sequences for any one series are given, those for the remaining series can be obtained by cyclic substitution of year numbers.

If a reduced design exists the groups of sequences to be allocated to the series can be determined by linking comparable sequences together in a chain, in the same way that blocks are linked together in the chain-block designs described, for example, by Cochran and Cox (1957, p. 463). Each chain so formed corresponds to a group of sequences. If all the sequences can be linked in a single chain no reduced design exists.

The maximum number of groups of sequences that can be formed in this way for a single test crop is given by the highest common factor h of the set of numbers d_j, representing the intervals (numbers of years between successive tests) of the test crop in each rotation. For example, the numbers d_j for the rice crop in Table 4 are 3 for rotation A, 1 and 3 for rotation B and 4 for rotation C. The value of h here is 1 so that no reduced design is available. When there are no phase differences, i.e. when the rotations do not include repeated test crops, the numbers d_j are simply the periods of the rotations.

The maximum number of groups that can be formed for two or more test crops simultaneously depends also on the relative positions of the test crops in the different rotations. Subject to the condition that every pair of test crops occupy the same positions relative to each other in every rotation in which they both occur, the number of groups is given by h', the highest common factor of the set of h for all the test crops. The following example illustrates the condition. If T and T' are test crops in a four-course rotation $TCCT'$, where C represents any other crop, then a six-course rotation including both test crops must be of the form (i) $TCCCCT'$ or (ii) $TCCT'CC$. In (i) T' immediately precedes T in both the four-course and six-course rotations; in (ii) T' is grown 3 years after T in both rotations. When this condition is not satisfied the number of groups is only a fraction of h'. A reduced design can always be constructed when h'', the highest common factor of the intervals between any tests, is greater than 1.

TABLE 5

Reduced design for a hypothetical experiment on rotations of differing lengths

Series Years I	*Series* Years II	*Rotations* A	B		C		
1, 13, ...	2, 14, ...	T_1	T_2	T''_2	T_3	T''_3	.
2, 14, ...	3, 15, ...	T'_1	T'_2	.	.	.	.
3, 15, ...	4, 16, ...	T_1	T''_2	T_2	.	T_3	T''_3
4, 16, ...	5, 17, ...	T'_1	.	T'_2	.	.	.
5, 17, ...	6, 18, ...	T_1	T_2	T''_2	T''_3	.	T_3
6, 18, ...	7, 19, ...	T'_1	T'_2	.	.	.	.
7, 19, ...	8, 20, ...	T_1	T''_2	T_2	T_3	T''_3	.
8, 20, ...	9, 21, ...	T'_1	.	T'_2	.	.	.
9, 21, ...	10, 22, ...	T_1	T_2	T''_2	.	T_3	T''_3
10, 22, ...	11, 23, ...	T'_1	T'_2	.	.	.	.
11, 23, ...	12, 24, ...	T_1	T''_2	T_2	T''_3	.	T_3
12, 24, ...	1, 13, ...	T'_1	.	T'_2	.	.	.

When the highest common factor of the periods of the rotations is greater than 1, but the other conditions are not met, it is worth considering whether a reduced design can be obtained by sacrificing information on minor contrasts between the selected test crops. Thus, if the tests of a test crop are divided into groups such that no contrasts within groups are to be confounded, it may be possible to construct a reduced design with the groups of tests formally regarded as different test crops.

The designs discussed in Section 1.3.1 are reduced designs. Only one reduced design can be constructed for the rotations of Table 3 because c is a prime number. Table 5 shows the cropping scheme for an experiment on three rotations, of lengths 2, 4 and 6 years. The arrangement is a reduced design without confounding of any comparisons if T, T' and T'' represent different test crops, or if T and T'' represent one test crop and T' another. It is also a reduced design if T' and T'' represent two confounded groups of tests for a single test crop and T represents either (a) a second test crop or (b) the same group as T''.

1.3.4. *Phase-confounded designs*

Even when reduced designs are not available, suitable designs with smaller blocks than the basic design can sometimes be obtained by partially confounding certain test-crop comparisons. Designs of this type will be called *phase-confounded designs*. Construction of these designs may require considerable skill to avoid undue complication in the analysis and, in some cases, actual loss of accuracy instead of the intended increase.

Some properties of phase-confounded designs will be discussed below with reference to a particular class of designs related to the reduced designs of Section 1.3.1. Designs of this class are available for any set of rotations with one or more test crops common to all rotations. They are constructed as follows:

(1) Select a particular test crop for which all comparisons except phase differences are to be determined as accurately as possible.
(2) Select one sequence for each rotation, so that all comparisons for the selected test crop, excluding phase differences, can be made in the same year just once per cycle of l years, where l is the lowest common multiple of the periods of the rotations. These sequences make up one series.
(3) The remaining series are obtained by cyclic substitution of year numbers. The total number of series required is therefore l.

In a design of this class no comparisons between the yields of the selected test crop in different rotations are completely confounded. Comparisons involving

TABLE 6

Cropping scheme in an alternative design for the Sudan Gezira combined rotations experiment (*Crowther and Cochran*, 1942)

Series I	II	III	...	XII	*Rotations* (1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Years														
1	2	3	...	12	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
2	3	4	...	1	C_1	F	D	F	D	F	F	D	F	F
3	4	5	...	2	C_1	C_2	C_3	F	F	D	F	F	D	F
4	5	6	...	3	C_1	F	D	C_4	C_5	C_6	F	F	F	D
5	6	7	...	4	C_1	C_2	C_3	F	D	F	C_7	C_8	C_9	C_{10}
6	7	8	...	5	C_1	F	D	F	F	D	F	D	F	F
⋮	⋮	⋮	...	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12	1	2	...	11	C_1	F	D	F	F	D	F	F	F	D

C: cotton. F: fallow. D: dura.

rotations of differing length are partially confounded; comparisons involving only rotations of equal length are not confounded at all. Comparisons for any other test crop may be partially or completely confounded, depending on the comparability of the sequences of any one series with respect to this other test crop. If the sequences fall into two or more groups, as defined in Section 1.3.3, some comparisons are completely confounded. If there is only one group no comparisons between different rotations are completely confounded but they may be partially confounded. All phase differences are completely confounded.

Table 6 shows the cropping scheme for one cycle of 12 years of a phase-confounded design of this class, with cotton as the selected test crop. The whole scheme is repeated identically in cycles of 12 years. All sequences are present but with unequal replication for the different rotations. Thus, the number of replicates of any sequence of a c-course rotation is $12/c$. Comparisons between cotton in rotations of equal length are not confounded; those between cotton in rotations of differing lengths are partially confounded (see Table 7). For dura, the other test crop, all comparisons are partially confounded including those between rotations of equal length.

This design will be compared with the basic design actually used in the experiment, which is described by Crowther and Cochran (1942). The number of plots per block in the basic design is 30, compared with the 10 plots per block of the phase-confounded design.

TABLE 7

Fractions of information confounded on (a) cotton means over 12 years, (b) annual cotton means in the design of Table 6

	(i)	(ii)	(iii)	(iv)	(v)
		Rotations included in the analysis			
Lengths of rotations compared	*All*	*All*	*All except* (1)	(1), (2), (4), (7)	(2), (4), (7)
	(a)	(b)	(b)	(b)	(b)
1, 2	0·06	0·19	—	0·21	—
1, 3	0·08	0·21	—	0·25	—
1, 4	0·08	0·20	—	0·25	—
2, 3	0·08	0·20	0·31	0·28	0·54
2, 4	0·04	0·08	0·11	0·13	0·21
3, 4	0·10	0·19	0·28	0·28	0·51

The fractions of information confounded on mean contrasts over 12 years between yields of cotton in rotations of differing lengths depend on the correlations between yield values recurring on the same plots. These correlations will be discussed in Section 2.2. Column (i) of Table 7 shows the confounded fractions for a fairly typical average correlation of 1/3. The fractions are reasonably small and it seems likely that in practice they would be more than offset by the reduced error variance associated with the smaller blocks of the design.

A different picture emerges, however, when contrasts between cotton yields in individual years are considered. Column (ii) of Table 7 shows the larger fractions confounded for these contrasts. The possibility must also be considered that the crop grown continuously may fail or give such poor yields that its results have to be omitted. In this case the fractions of information confounded on contrasts in individual years between the remaining rotations are increased by about 50 per cent., as shown in column (iii) of Table 7.

For other designs of the same type the fractions confounded may be even larger, particularly when the rotations all differ in length. Columns (iv) and (v) of Table 7 give the fractions for cotton corresponding to columns (ii) and (iii) for a design with only the four rotations (1), (2), (4) and (7) of Table 6. In general, it appears that the confounded fractions are largest for contrasts between rotations whose periods have no common factor greater than 1. Thus in all columns of Table 7 the fractions are smallest for the comparison of rotations of lengths 2 and 4 years.

Other points to be considered in assessing the value of the design of Table 6 are that (a) comparisons for dura, the other test crop, are partially confounded, (b) the unequal replication of sequences for different rotations results in contrasts between rotations of short length being estimated with the greatest accuracy, and (c) the total number of plots required is large. The fractions of information confounded on dura are larger than for cotton, particularly in comparisons between rotations of the same length. Point (b) is illustrated in Table 8, which gives efficiency factors, relative to the basic design, for differences between cotton yields in individual years. Phase-confounded designs invariably require a larger minimum number of plots than a basic or reduced design. Thus, the design of Table 6 requires at least 120 plots; basic designs with any multiple of 30 plots are available for the same set of rotations.

TABLE 8

Efficiency factors for cotton in individual years of the design of Table 6 relative to a basic design

(i)	(ii)	(i)	(ii)	(i)	(ii)
1, 2	1·63	2, 2	1·50	3, 3	1·00
1, 3	1·19	2, 3	0·96	3, 4	0·69
1, 4	0·97	2, 4	0·92	4, 4	0·75

(i) lengths of rotations compared; (ii) efficiency factors.

Other types of phase-confounded design have still to be classified and studied. Some types, such as that for the experiment described by Stevens (1956), appear to be reasonably efficient and capable of relatively simple analysis.

Proposals are common for long-term rotation experiments that include large numbers of rotations of varying lengths, often with only minor differences between some of the rotations. Critical examination of the proposals usually shows that the requirements of the experimenters can be met by a two-stage programme of work. In the first stage the short-term effects of the rotations are compared by experiments of the type discussed in Section 1.2, or other short-term experiments. The results of these experiments are then used to select a small number of contrasting rotations for comparison in one or more long-term experiments. In making the selection it is often possible to arrange that an efficient basic or reduced design is used.

1.4. *Auxiliary Treatments*

Auxiliary fertilizer and other treatments are very important in rotation experiments. Suitably chosen, they provide (a) comparisons between the rotations and other methods of controlling fertility, (b) a method for testing hypotheses about the underlying causes of the effects of rotations and (c) an extended range of conditions for the experiments.

Auxiliary treatments are allocated to particular crops. The different treatments of a sequence of crops are usually combined factorially. Repeated tests of the same treatments can also be combined factorially to give independent tests in different years of application. A treatment whose long-term effects require to be estimated must, of course, be applied cumulatively, i.e. on the same plots in each year of application. Extensive use can be made of split-plot designs, confounding and fractional replication.

The design of schemes for auxiliary treatments in rotation experiments raises problems that are not met in annual experiments on similar treatments. A full discussion is outside the scope of the present paper, but it is worth noting that auxiliary treatments can be introduced more readily and efficiently into basic or reduced designs, particularly those of Section 1.3.1, than into designs with complex systems of phase confounding. The greatest flexibility of design is obtained when the numbers of levels of all treatment factors, numbers of replicates and numbers of sub-plots are all powers of 2. Principles and methods used to allocate auxiliary treatments to the experiment of Table 2 will be described in detail elsewhere.

2. The Analysis of Cyclic Rotation Experiments

2.1. *Introduction*

The remainder of this paper deals with some problems of statistical technique arising in the analysis of cyclic rotation experiments. The different crops of a rotation experiment must normally be analysed separately because the effects of rotations vary considerably from crop to crop. An investigation of these variations over a range of contrasting crops may provide valuable information on the different ways rotations operate. An alternative approach, which is sometimes suggested, is to analyse a single variate, such as economic value, that is common to all crops. This type of analysis and its interpretation are made extremely complicated by variations in the relative behaviour of different crops from year to year, plot to plot and block to block. If the assessment of economic value is the main object, a better procedure may be to do a separate analysis for each crop and then combine the results, having ensured in the design that all crops are capable of giving meaningful economic comparisons.

The first stage in the analysis of the yields of a single test crop consists of determining, for each year of the experiment, the mean yields of the test crop in the different rotations and, where necessary, the different phases of the same rotation. Thus there is one mean yield per year for each test of the test crop. Differences and other contrasts between the test means in any year estimate the effects of the rotations on the test crop in that year.

Some of the points arising in the analysis of a single year's results will be illustrated by examples of the designs discussed in Section 1 of the paper. The yields of potatoes in 1 year of the experiment of Table 2 form a single 2×4 table of replicates × tests. Because there is just one test of potatoes per rotation the table could also be described as a replicates × rotations table in this particular case. In general, however, the terms "tests" and "rotations" cannot be used synonymously. The column means of the table give the required test means and the interaction gives 3 d.f. for experimental error. The wheat and barley results can be analysed in the same way.

The analysis of the yields of barley in 1 year of the experiment of Table 3 can be based on a single table as before, but two tables of replicates × tests are required for potatoes, one for replicates × P_1 *v.* P_2 *v.* P_3 *v.* P_4 and the other for replicates × P'_3 *v.* P'_4. These tables are kept separate for the estimation and comparison of test means because the contrast between the two groups of tests is confounded. The error degrees of freedom in the two tables can be pooled, however, to give estimates of the errors of comparisons between test means within either group.

For the analysis of rice in 1 year of the experiment of Table 4 (or any crop in a basic design) only one table of replicates × tests is required because no comparisons between test yields are confounded. The table for this experiment differs from those previously considered in that it includes two columns for one of the rotations. Comparisons between the column (test) means of the table provide an assessment not only of the relative yields of cotton in the three rotations but also of the relative yields of cotton 1 and 2 years after grass in rotation B.

A more elaborate technique is required for the analysis of cotton yields in 1 year of the experiment of Table 6 as the ordinary test means are not orthogonal with blocks. Adjustment of the test means for block differences is complicated by the varying numbers of test yields in the different blocks. The method described by Tocher (1952) for incomplete block experiments with unequal block sizes can be used. As the total number of plots is the same for each block, Tocher's reservations concerning possible heterogeneous errors are not relevant in this particular application.

The test means for one crop over a period of years can be set out conveniently in 1 or more years × tests tables, one table for each group of tests. Each years × tests table is complete (apart from accidental missing values) for any experiment including all phases of each rotation.

The present discussion is concerned with the analysis of a single complete years × tests table of a replicated experiment with basic or reduced design. We will not deal with (a) the comparison of tests in different groups, (b) the combination of results for different test crops, (c) experiments of phase-confounded design, or (d) non-replicated experiments.

The conventional method of summarizing a years × tests table consists of (1) estimating the mean effects over the years, and (2) fitting polynomial curves, one for each column of the table, representing the relationships between annual test means and time. This is the method used by Crowther and Cochran (1942) for the analysis of two rotation experiments and by Cochran (1939) and others for the analysis of long-term experiments generally.

The aim of (2) is to determine whether the rotation effects are changing cumulatively, and if so to estimate the extent of the changes. Trends can, however, arise from changes in external factors that influence the crops as well as from accumulation of the effects of the rotations. Among these external factors may be mentioned meteorological conditions, changes in farming methods and the incidence of pests and diseases due to external causes. A common feature of many long-term experiments in the past 30 years has been the presence of trends due to improvements in farm management. Trends in external factors can be allowed for, in part at least, by multiple regression in stage (2) above on suitable variates representing the external factors, provided such variates are available. Even when the variates for external factors are not subject to trends, multiple regression may be useful (i) for the increased accuracy resulting from reduction in the residual year-to-year variations, and (ii) because the relationships may throw some light on the way the effects of the rotations are produced.

Cumulative effects may eventually tend to limiting values, called limiting effects, if the experiment is sufficiently long. If they exist, limiting effects can be estimated by differences between mean yields over a period of years after stability is reached. They can be used to assess the ultimate relative values in practice of different cropping systems. Definitions of cumulative, limiting and other types of effects in long-term experiments are given by Yates (1949).

Methods for estimating the means, trends and regressions of a single years × tests table are discussed in the following Sections. Years, tests, groups of plots carrying the same sequences of crops (excluding groups of plots not carrying the test crop in any of the years under consideration), and replicates will be numbered as follows:

Years:	$i = 1, 2, \ldots, N$,	Groups of plots:	$k = 1, 2, \ldots, p$,
Tests:	$j = 1, 2, \ldots, t$,	Replicates:	$l = 1, 2, \ldots, r$.

When necessary a particular plot will be labelled (k, l). It will be supposed that the N years exclude all preliminary years. Some simplification of analysis is often possible when N is a multiple of the period of the complete experiment cycle, but unless otherwise stated the results given below are appropriate for any number of years.

The mean yield for test j in year i will be denoted by $\bar{y}_{ij}$, and the seasonal and trend variates by x_{im} ($m = 1, 2, \ldots, q$, and $q < N$). Also $\bar{x}_m$ is the mean value for variate m over the N years and

$$x'_{im} = x_{im} - \bar{x}_m.$$

The true mean yield for test j in year i is represented by τ_{ij}, τ_j is the mean of the τ_{ij} over a hypothetical infinite population of years with mean values of the x_{im} given by $\bar{x}_m$, and

$$\bar{\tau}_j = N^{-1} \sum_i \tau_{ij}.$$

The true regressions of $\bar{y}_{ij}$ on x_{im} are denoted by β_{jm}. Formally $\bar{y}_{ij}$ can then be represented by

$$\left.\begin{aligned} \bar{y}_{ij} &= \tau_{ij} + \epsilon_{ij} \\ &= \tau_j + \sum_m \beta_{jm} x'_{im} + \psi_{ij} + \epsilon_{ij}, \end{aligned}\right\} \qquad (2.1.1)$$

or by

$$\bar{y}_{ij} = \bar{\tau}_j + \sum_m \beta_{jm} x'_{im} + \psi'_{ij} + \epsilon_{ij}. \qquad (2.1.2)$$

The ψ_{ij} represent residual seasonal variations in the yields and effects and

$$\psi'_{ij} = \psi_{ij} - N^{-1} \sum_i \psi_{ij}. \qquad (2.1.3)$$

The ϵ_{ij} are experimental errors due to differences between plots and changes in plot effects from year to year.

Equations (2.1.1) and (2.1.2) will be referred to as equations for models I and II respectively. Model I can be used when the experimenter's primary purpose is to select a suitable rotation for practical farming and inferences are required about the effects of rotations over a range of seasonal conditions. The estimates of τ_j can be replaced by estimates of $\tau_j + \Sigma \beta_{jm}(x_m - \bar{x}_m)$ to adjust for differences between known average seasonal conditions, represented by x_m, and those obtained in the experiment. It should be recognized that (a) with this model only variates that can be predicted or controlled should be included in the regression analysis, and (b) experiments should be done over a range of sites as well as a range of years to obtain complete generality.

An analysis based on model II refers only to the particular conditions obtained in the experiment. It is appropriate when the different types of effect produced by the rotations require to be separated and explained. Whatever the primary purpose of the experiment no opportunity should be lost to study fundamental questions, because rotation experiments are expensive in both time and money.

The analysis will be developed in terms of model I, as this is a little easier than model II to deal with algebraically. Theoretical results for the two models differ only in the expressions for the variances of the estimates of the mean rotation effects; these differences will be indicated where necessary.

At first sight a straightforward multiple regression technique appears to be required, but the analysis is complicated by (1) correlations between ϵ_{ij} and $\epsilon_{i'j'}$, where i' and j' are such that test j' occurs on the same plots in year i' as test j in year i, and (2) heterogeneity of the ψ_{ij}. Special methods are required to obtain unbiased estimates of error. Also the ordinary estimates of τ_j and β_{jm} may be inefficient, as a result of (1).

In Section 2.2 the types of variation occurring in ψ_{ij} and ϵ_{ij} are discussed and specified in terms of components of variance. Sections 2.3, 2.4 and 2.5 deal with the estimation of these components and their use in evaluating the errors of the ordinary estimates of τ_j and β_{jm}. Methods for using the components of variance to obtain more accurate estimates of τ_j and β_{jm} are then discussed in Sections 2.6 and 2.7.

2.2. *Components of Variance*

Correlations between the errors of yield values of annual crops recurring on the same plots in long-term experiments are generally small and positive. They rarely exceed about 0·5 and are often much smaller. Although small, the correlations are important, particularly in determining the errors of mean yields over the years. If they were ignored, the variance of the mean of n yield values from the same plot in n different years would be underestimated by a factor of $(n-1)\rho+1$, where ρ is the mean correlation. Differential trends of plots within blocks, which would otherwise be important in determining the errors of estimated trends, have usually been found to be small, i.e. the correlations are persistent (see, for example, Cochran, 1939, and Patterson, 1953). By contrast, the correlations between two yield values, y_i and $y_{i'}$ say, recurring on the same plot of a long-term experiment on a perennial crop, tend to be large for small values of $|i-i'|$ and to fall off with increasing $|i-i'|$, i.e. differential plot trends are important. We are not, however, concerned with perennial crops in the present paper.

It appears, therefore, that the correlations occurring in rotation experiments with annual crops often can be adequately allowed for by regarding the experimental errors of yield values as the sum of two parts, of which one is constant over the years with variance σ_p^2 from plot to plot and the other is independent from year to year with variance σ_w^2. Only this approach, which has also been adopted by Crowther and Cochran (1942) and Yates (1954), will be considered in the present paper.

The variations in ψ_{ij} can also be represented in terms of components of variance and covariance. We define

$$\operatorname{cov}(\psi_{ij}, \psi_{i'h}) = \begin{cases} \alpha_{jh} & (i = i'), \\ 0 & (i \neq i'), \end{cases} \qquad (2.2.1)$$

for all pairs of values $j, h = 1, 2, \ldots, t$. For model II

$$\operatorname{cov}(\psi'_{ij}, \psi'_{i'h}) = \begin{cases} \dfrac{(N-1)\,\alpha_{jh}}{N} & (i = i'), \\[2ex] \dfrac{-\alpha_{jh}}{N} & (i \neq i'). \end{cases} \qquad (2.2.2)$$

The α_{jh} include components of the mean seasonal differences between the $\bar{y}_{ij}$. If these components are to be excluded from the analysis the α_{jh} can be replaced by α'_{jh}, where

$$\alpha'_{jh} = \alpha_{jh} - \frac{\sum_h \alpha_{jh}}{t} - \frac{\sum_j \alpha_{jh}}{t} + \frac{\sum_{j,h} \alpha_{jh}}{t^2}. \tag{2.2.3}$$

It is convenient also to define

$$\sigma^2_{ty} = \frac{t \sum_j \alpha_{jj} - \sum_{j,h} \alpha_{jh}}{t(t-1)} = \frac{\sum_j \alpha'_{jj}}{(t-1)}, \tag{2.2.4}$$

a component of variance representing the average residual years × tests variations. This component will be said to be homogeneous if the α'_{jj} are the same for all j and the α'_{jh} are the same for all $j \neq h$. This implies that

$$\sum_j l_{js} \bar{y}_{ij} \quad \text{and} \quad \sum_j l_{js'} \bar{y}_{ij},$$

two orthogonal contrasts such that

$$\sum_j l_{js} = \sum_j l_{js'} = \sum_j l_{js} l_{js'} = 0,$$

vary independently from year to year and is, therefore, often unrealistic.

Components of variance σ^2_{tys} can also be defined for particular contrasts as follows:

$$\sigma^2_{tys} = \frac{\sum_{j,h} l_{js} l_{hs} \alpha_{jh}}{\sum_j l^2_{js}}. \tag{2.2.5}$$

2.3. *Estimation of σ^2_p and σ^2_w: Rotations of Equal Length*

The method of estimating σ^2_p and σ^2_w for a particular test crop of an experiment of basic or reduced design depends on (i) whether the rotations with the crop are of equal or unequal length, and (ii) the number of groups of tests of the test crop that can be formed such that contrasts within groups are not confounded and contrasts between groups are confounded.

The analysis is particularly simple when the rotations are of equal length, the sequences are arranged in c series and there is just one group of tests, e.g. for the wheat test crop of Table 2. When $N \leqslant c$ all variances can be expressed in terms of $\sigma^2_p + \sigma^2_w$, which can be estimated directly from the total error sum of squares over the years. When $N > c$, σ^2_p and σ^2_w are estimated by partitioning the total error sum of squares into two components, a plot error sum of squares derived from plot totals and a plot × year error sum of squares. Essentially the same method is used by Cochran (1939), Patterson (1953) and Yates (1954) for fixed-rotation experiments. It is appropriate whether the cycles of the experiment are complete or not.

There are $(t-1)(r-1)$ degrees of freedom for error in any 1 year, making a total of $N(t-1)(r-1)$ degrees of freedom over the years. For each series a two-way replicates × tests table of plot totals is formed and the replicates × tests sum of squares calculated with appropriate divisors for squared plot totals. The total of these sums of squares over the c series is the plot error sum of squares with $c(t-1)(r-1)$ degrees of freedom; the plot × year error sum of squares is obtained by subtraction. The expected values of the mean squares are then as follows:

	d.f.	M.S.	$\mathscr{E}$(M.S.)
Plot error	$c(t-1)(r-1)$	M_p	$N\sigma_p^2/c+\sigma_w^2$
Plot × year error	$(N-c)(t-1)(r-1)$	M_w	σ_w^2
Total error	$N(t-1)(r-1)$		$\sigma_p^2+\sigma_w^2$

The estimates of σ_p^2 and σ_w^2 are, therefore, $c(M_p-M_w)/N$ and M_w respectively. The method is efficient when N is a multiple of c and otherwise slightly inefficient.

The above analysis can also be used for an experiment of similar design but with the plots divided into fewer than c series on the ground. In this case the plots of each actual series are grouped in two or more pseudo-series for the purpose of analysis. Plots in the same pseudo-series carry crop sequences that are comparable with respect to the test crop being analysed; plots in different pseudo-series do not (Section 1.3.3). The total number of pseudo-series is then c.

For a test crop with two or more groups of tests the analysis is in general more complicated. If the different groups of tests have no plots in common, separate analysis can be done for each group, and σ_p^2 and σ_w^2 estimated from pooled plot error and plot × year error mean squares. If exactly the same plots are used in different years for the different groups of tests a modified form of the above analysis can be used. Suppose, for example, that the potatoes test crop in the experiment of Table 2 is replaced by wheat so that there are altogether eight tests of wheat in two groups of four. The total number of error degrees of freedom and the coefficient of σ_p^2 in the expected value of the plot error mean square are then doubled, but the number of degrees of freedom for plot error remains the same. If some but not all the plots are common to each group of tests the analysis of Section 2.4 is required (see in particular Section 2.4.4). The analysis of potatoes in the experiment of Table 3 provides an example of this situation. Here P_1 and P_2 are on plots which do not also have either test of the group P_3', P_4'.

2.4. *General Method for Estimating σ_p^2 and σ_w^2 in Basic or Reduced Designs*

An analysis of variance separating the total error sum of squares into plot and plot × year components can also be used to estimate σ_p^2 and σ_w^2 for basic or reduced designs with rotations of unequal length. Non-orthogonality of the plot totals and totals for block-year combinations complicates the analysis and results in some loss of efficiency in the estimation of σ_p^2 and σ_w^2; such non-orthogonality occurs if the numbers d_j, defined in Section 1.3.3, are not all equal. Yates (1954) gives full details of the analysis for one complete cycle (12 years) of the rice–pasture experiment of Table 4. The method is also appropriate for other numbers of years.

The general form of the analysis-of-variance method (the method followed by Yates) will be discussed in Sections 2.4.1 to 2.4.3. In this form the method is suitable for any basic or reduced design with no contrasts confounded in the test crop being analysed. In Sections 2.4.5 several other methods that are not fully efficient will be considered and compared with the analysis-of-variance method in terms of efficiency and convenience, with particular reference to the rice-pasture experiment.

2.4.1. *Analysis-of-variance method*

The analysis-of-variance method consists of (1) calculating the total sum of squares for error over all years, (2) fitting constants for plots, eliminating blocks, blocks × years and the effects of rotations, so as to minimize the residual plot × year sum of squares and (3) equating the plot and plot × year error mean squares of the resulting analysis of variance to their expected values.

Stage (1) presents no special problems. Stages (2) and (3) can be done in several ways of which the direct procedure of solving the normal equations will be considered first.

This consists of fitting constants for plots, eliminating the year effects on blocks, but ignoring the effects of the rotations, to the yield values of each of the r years × plots tables, one table for each replicate. The r sets of constants fitted in this way can be arranged in a $p \times r$ table of p_{kl}, where p_{kl} is the constant for plot (k, l).

Differences between the rows of this table are confounded with the effects of the rotations, and contrasts between the columns involve block effects. An interaction contrast, e.g. $p_{11}-p_{21}-p_{12}+p_{22}$, either gives the required plot contrast or is equal to zero due to redundancies among the constants. There is therefore a maximum of $(p-1)(r-1)$ d.f. for plot error. The identification of redundancies is discussed below.

An $N \times p$ matrix, in which element (i, k) is a unit element if plots k carry the test crop in year i and zero otherwise, is required in the process of fitting the plot constants. This matrix will be denoted by $\mathbf{G}$ and called the years × plots incidence matrix.

We also define P_{kl}, the total for plot (k, l), and B_{kl}, the sum of the totals for replicate l over all years in which plot (k, l) carries the test crop. If $\mathbf{B}_l$ is the matrix of year totals for replicate l then B_{kl} is element k of the $p \times 1$ matrix $\mathbf{G}'\mathbf{B}_l$. The elements, P'_{kl}, of a $p \times 1$ matrix, $\mathbf{P}_l$, of adjusted plot totals are given by

$$P'_{kl} = P_{kl} - B_{kl}/t. \tag{2.4.1}$$

The required constants p_{kl} are then the elements of $\mathbf{p}_l$, where

$$\boldsymbol{\phi}\mathbf{p}_l = \mathbf{P}_l. \tag{2.4.2}$$

Here $\boldsymbol{\phi}$ is a $p \times p$ symmetric matrix given by

$$\boldsymbol{\phi} = (\mathbf{1}'\mathbf{G})^{\delta} - \mathbf{G}'(\mathbf{G}\mathbf{1})^{-\delta}\,\mathbf{G}, \tag{2.4.3}$$

where $(\mathbf{1}'\mathbf{G})^{\delta}$ is the diagonal matrix of the column totals of $\mathbf{G}$ and $(\mathbf{G}\mathbf{1})^{-\delta} = t^{-1}\mathbf{I}_N$ is the diagonal matrix of the reciprocals of the row totals of $\mathbf{G}$, following the notation used by Tocher (1952).

Equation 2.4.2 has no unique solution for $\mathbf{p}_l$ because $\boldsymbol{\phi}$ is singular. The rank of $\boldsymbol{\phi}$ is related to the number of degrees of freedom for plot error and will be denoted by ν_p. If $\nu_p = p-1$ a solution with the restriction

$$\mathbf{1}'\mathbf{p}_l = 0 \tag{2.4.4}$$

can be obtained from

$$\mathbf{p}_l = (\boldsymbol{\phi} + g\mathbf{1}\mathbf{1}')^{-1}\,\mathbf{P}_l = \boldsymbol{\Lambda}\mathbf{P}_l, \tag{2.4.5}$$

where $\mathbf{1}$ is the $p \times 1$ matrix of unit elements and g is any convenient scalar. If $\nu_p < p-1$ the equations (2.4.2) fall into $p-\nu_p$ independent groups, each of which can be solved separately. (See also Section 2.4.4.)

The value of ν_p can be determined readily from $\mathbf{G}$. Any two columns of $\mathbf{G}$ are said to be linked if they have unit elements in the same row at least once. The columns can be divided into one or more groups such that all columns in the same group are linked and all columns in different groups are not linked. The number of groups so formed is $p-\nu_p$. The minimum possible number of groups of columns is equal to the number of groups of comparable sequences discussed in Section 1.3.3. Thus the maximum rank of **G** is $p-h$. The rank of **G** may be less than maximum when the number of years is small.

The plot error sum of squares is

$$\sum_l \mathbf{p}_l' \mathbf{P}_l - \frac{1}{r} \sum_l \mathbf{p}_l' \sum_l \mathbf{P}_l \tag{2.4.6}$$

and the plot × year error sum of squares is obtained by subtracting the plot error sum of squares from the total error sum of squares of stage (1). The expected values of the error mean squares are as follows:

	d.f.	$\mathscr{E}$(M.S.)
Plot error	$(r-1)\nu_p$	$\dfrac{N(t-1)}{\nu_p}\sigma_p^2+\sigma_w^2$
Plot × year error	$(r-1)\nu_w$	σ_w^2
Total	$N(r-1)(t-1)$	$\sigma_p^2+\sigma_w^2$

Here ν_w is defined as $N(t-1)-\nu_p$.

2.4.2. *Yates's method of orthogonal functions*

Yates (1954) used an alternative method for determining the plot error sum of squares. The method is to construct a set of $t-1$ orthogonal functions of the adjusted plot totals for each replicate:

$$\sum_k a_{ks} P_{kl}' = \sum_k a_{ks}' p_{kl} \quad (s = 1, 2, \ldots, t-1) \tag{2.4.7}$$

such that

$$\left.\begin{aligned} \sum_k a_{ks} a_{ks'}' &= 0 \quad \text{for all } s \neq s', \\ &\neq 0 \quad \text{for} \quad\;\; s = s'. \end{aligned}\right\} \tag{2.4.8}$$

The sum of squares of deviations between replicates,

$$\text{dev}^2\left(\sum_k a_{ks} P_k'\right) = D_s, \tag{2.4.9}$$

then has expected value

$$(r-1)\left(\sum_k a_{ks}'^2 \sigma_p^2 + \sum_k a_{ks} a_{ks}' \sigma_w^2\right) \tag{2.4.10}$$

so that the plot error sum of squares is

$$\sum_s \left(\frac{D_s}{\sum_k a_{ks} a_{ks}'}\right). \tag{2.4.11}$$

This method has the advantage that the matrix inversion of equation (2.4.5) is avoided. Redundancies are automatically identified in the process of constructing the orthogonal functions. Over a complete cycle of the experiment suitable orthogonal functions can be constructed fairly readily owing to the considerable degree of balance in the years × plots incidence matrix. Yates (1954) pointed out, however, that the method may become excessively complicated if there is an incomplete experimental cycle. For full details of the procedure for the rice–pasture experiment reference should be made to Yates (1954).

2.4.3. *Method of successive approximations*

The plot error sum of squares can also be obtained using the iterative method for fitting constants described by Stevens (1948); see also Yates (1952) and Yates (1960, Section 5.24). The constants are fitted to (1) r tables of which the lth is the years × plots incidence matrix with the deviations of the plot and year means for replicate l from the general mean of the replicate in the appropriate margins, and (2) an additional table consisting of the sums of the values in the other tables. The sum of squares for plot constants fitted to the table for replicate l provides a numerical value for $\mathbf{p}_l'\mathbf{P}_l$ in equation (2.4.6) and the corresponding sum of squares from the table of totals is used to determine $\Sigma\mathbf{p}_l'\Sigma\mathbf{P}_l$. This method is particularly suitable for an electronic computer because no preparation of normal equations is required. No modifications are required when there are redundant constants. If there are only two replicates some computation can be saved by basing the analysis on differences between corresponding yield values in the two replicates. The one disadvantage of Stevens's method, that the variances of the constants cannot be determined readily, is of no consequence in this application as the variances are not required.

2.4.4. *Analysis-of-variance method for reduced designs*

It was pointed out in Section 2.4.1 that equations (2.4.3) fall into two or more independent groups when the rank of $\boldsymbol{\phi}$ is less than $p-1$. This will always be the case for a reduced design. The groups of equations correspond to the series of the reduced design. Each series can be analysed separately and σ_p^2 and σ_w^2 estimated from the pooled sums of squares for plot error and plot × year error over the series. In the analysis of a particular series the matrix $\mathbf{G}$ is replaced by the appropriate sub-matrix $\mathbf{G}_1, \mathbf{G}_2, \ldots$ of $\mathbf{G}$ giving the years × plots incidence matrix for the plots of that series. Correspondingly $\boldsymbol{\phi}$ is replaced by $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots$ as appropriate. Although the matrices $\mathbf{G}_1, \mathbf{G}_2, \ldots$ are different, the matrices $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots$ are the same for each series when the cycles of the experiment are complete.

When there are two or more groups of tests, i.e. some comparisons for the test crop being analysed are confounded, each series must be analysed separately to eliminate block and block × year effects. In this case also, the analysis simplifies when the cycles of the experiment are complete.

2.4.5. *Other estimates of* σ_p^2 *and* σ_w^2

The estimates of σ_p^2 and σ_w^2 given by the analysis-of-variance method are particular cases of a general class of estimates which can be obtained from the plot constants and adjusted plot totals. The general procedure is to equate

$$S = \sum_l \mathbf{P}_l' \mathbf{U}\mathbf{P}_l - \frac{1}{r}\Big(\sum_l \mathbf{P}_l'\Big)\mathbf{U}\Big(\sum_l \mathbf{P}_l\Big) \quad (l = 1, 2, \ldots, r), \tag{2.4.12}$$

and either the total error sum of squares or the plot × year error sum of squares to their expected values, and solve the resulting equations. Here $\mathbf{U}$ is a symmetric $p \times p$ matrix. The expected value of S is

$$\mathscr{E}(S) = (r-1)\,\mathrm{tr}\,(\mathbf{U}\boldsymbol{\phi}^2\sigma_p^2 + \mathbf{U}\boldsymbol{\phi}\sigma_w^2). \tag{2.4.13}$$

The estimates obtained depend on the choice of $\mathbf{U}$. The following particular cases will be compared in terms of efficiency and ease of estimation:

I. $\mathbf{U} = \boldsymbol{\Lambda}$, the inverse matrix defined in equation (2.4.5). The estimates obtained are those of the analysis-of-variance method.
II. $\mathbf{U} = \boldsymbol{\Lambda}^2$, with σ_w^2 estimated as in case I.
III. $\mathbf{U} = \mathbf{I}$, with S and the total error sum of squares used to estimate σ_p^2 and σ_w^2.
IV. As in III but with $\mathbf{U} = \mathbf{u}^\delta$, a diagonal matrix whose elements are the reciprocals of the diagonal elements of $\boldsymbol{\phi}$.

The expression (2.4.13) can be simplified in cases I and II by noting that $\boldsymbol{\Lambda}\boldsymbol{\phi}^2 = \boldsymbol{\phi}$, $\boldsymbol{\Lambda}^2\boldsymbol{\phi} = \boldsymbol{\Lambda}$, and that $\operatorname{tr}\boldsymbol{\Lambda}\boldsymbol{\phi}$ and $\operatorname{tr}\boldsymbol{\Lambda}^2\boldsymbol{\phi}^2$ are each equal to the rank of $\boldsymbol{\phi}$.

The efficiencies of the estimates obtained in the above cases can be compared fairly readily for any particular years × plots incidence matrix if it is supposed that the plot and plot × year components of error are independently distributed normal deviates. Under these conditions it can be shown that

(i) $$\operatorname{var} S = 2(r-1)\operatorname{tr}\{(\mathbf{U}\boldsymbol{\phi}^2)^2\sigma_p^4 + 2\mathbf{U}\boldsymbol{\phi}\mathbf{U}\boldsymbol{\phi}^2\sigma_p^2\sigma_w^2 + (\mathbf{U}\boldsymbol{\phi})^2\sigma_w^4\}, \qquad (2.4.14)$$

(ii) the covariance of S with the total error sum of squares is

$$2(r-1)\operatorname{tr}\{\mathbf{U}\boldsymbol{\phi}^3\sigma_p^4 + 2\mathbf{U}\boldsymbol{\phi}^2\sigma_p^2\sigma_w^2 + \mathbf{U}\boldsymbol{\phi}\sigma_w^4\} \qquad (2.4.15)$$

and

(iii) S is independent of the plot × year error mean square of the analysis of variance.

These results can be proved either by repeated use of a lemma given by Grundy (1950) or by using the techniques described by Searle (1956, 1958). Grundy's lemma states that if z_1, z_2, z_3 and z_4 are jointly normal deviates then

$$\operatorname{cov}(z_1 z_2, z_3 z_4) = \operatorname{cov}(z_1, z_3)\operatorname{cov}(z_2, z_4) + \operatorname{cov}(z_1, z_4)\operatorname{cov}(z_2, z_3). \qquad (2.4.16)$$

Fully efficient estimates are not readily available, but large-sample formulae for the variances and covariances of the most efficient estimates of σ_p^2 and σ_w^2 that can be obtained from the experimental error degrees of freedom are given by the inverse of the following matrix:

$$\frac{r-1}{2}\begin{bmatrix} \sum_k \frac{\eta_k^2}{(\eta_k\sigma_p^2+\sigma_w^2)^2} & \sum_k \frac{\eta_k}{(\eta_k\sigma_p^2+\sigma_w^2)^2} \\ \sum_k \frac{\eta_k}{(\eta_k\sigma_p^2+\sigma_w^2)^2} & \sum_k \frac{1}{(\eta_k\sigma_p^2+\sigma_w^2)^2} + \frac{\nu_w}{\sigma_w^4} \end{bmatrix}, \qquad (2.4.17)$$

where the η_k are the latent roots of $\boldsymbol{\phi}$. The expressions for the elements of this matrix can be obtained by a straightforward application of the maximum-likelihood method.

The efficiencies of linear functions of the estimates of σ_p^2 and σ_w^2 in methods I to IV can therefore be defined as the ratios of the asymptotic variances given by (2.4.17) to the variances obtained using equations (2.4.14) and (2.4.15). This definition ignores the information on σ_p^2 and σ_w^2, usually small in amount, which could be obtained from block × year contrasts.

Table 9 shows the percentage efficiencies of estimates of various linear functions of σ_p^2 and σ_w^2 for one cycle of 12 years of the rice–pasture experiment of Table 4. The efficiencies are given for varying values of σ_p^2/σ_w^2 mainly concentrated in the range

TABLE 9

Percentage efficiencies of estimates of linear functions of σ_p^2 and σ_w^2 in the rice–pasture rotation experiment (Table 4) by (I) the analysis-of-variance method and (II), (III), (IV) alternative methods

σ_p^2/σ_w^2	(a) σ_p^2				(b) σ_w^2			(c) $\frac{1}{2}\sigma_p^2+\sigma_w^2$			(d) $\sigma_p^2+\sigma_w^2$	(e) $2\sigma_p^2+\sigma_w^2$			(f) $4\sigma_p^2+\sigma_w^2$			(g) $6\sigma_p^2+\sigma_w^2$		
	I	II	III	IV	I	III	IV	I	III	IV	I, III, IV	I	III	IV	I	III	IV	I	III	IV
0	83	53	100	92	95	100	98	98	100	99	100	96	100	98	87	100	94	85	100	93
0·2	99	77	91	96	98	96	98	99	99	100	100	100	96	99	100	93	98	99	92	97
0·4	100	87	83	93	99	87	95	99	98	99	99	99	92	96	100	87	95	100	86	94
0·6	99	92	78	90	99	77	91	98	98	99	98	98	89	94	98	83	92	98	81	91
0·8	98	94	74	87	100	67	86	97	98	98	97	97	87	92	97	80	90	97	78	89
1·0	96	96	72	85	100	58	80	96	98	98	96	96	85	91	96	78	88	96	76	87
2·0	93	99	67	81	100	28	55	93	98	97	92	92	80	87	93	73	84	93	71	83
∞	87	100	60	74	100	0	0	87	98	93	87	87	73	81	87	66	78	87	64	76

$\sigma_p^2/\sigma_w^2 = 0$ to 1 most likely to occur in practice. Moderate losses of efficiency are of little consequence when large numbers of error contrasts are available.

For this design at least, the analysis-of-variance method I is the most consistently reliable of the four considered. In view of the high efficiency obtained there is little point in attempting the much more elaborate calculations required for the maximum-likelihood method. In some circumstances the estimates of II, III and IV can be more efficient than those of I but their overall performance is less satisfactory. In particular the estimate of σ_p^2 in method II has little to recommend it, since its efficiency is greatest for large values of σ_p^2/σ_w^2. (In fact the estimate tends to be fully efficient as σ_p^2/σ_w^2 tends to ∞.) Moreover, method II requires more computation than I as the vectors $\mathbf{p}_l$ and the matrix $\mathbf{\Lambda}$ have to be determined explicitly. The estimates of III and IV are simpler to calculate than those of I and may therefore be useful when computational facilities are limited. The estimates of III tend to be fully efficient when $\sigma_p^2/\sigma_w^2 \to 0$ but their efficiencies fall fairly rapidly with increasing σ_p^2/σ_w^2, except for linear functions with the coefficients of σ_p^2 between about $\frac{1}{2}$ and 1 times the coefficients of σ_w^2. The estimate of $\sigma_p^2+\sigma_w^2$ is the same in methods I, III and IV; it is in fact the pooled error mean square of the analysis of variance.

Qualitatively similar results are obtained with other designs. These will not be discussed here beyond noting that the superiority of the analysis-of-variance method over the others, particularly II and III, is even more marked when the numbers of values making up the plot totals differ widely.

Other methods of estimating σ_p^2 and σ_w^2 using mean squares involving components of both the plot and block $\times$ year variations need not be considered here. These methods usually require much less computation than the methods discussed above, but unless the block $\times$ year variations are small (and this is unlikely in practice) they are often inaccurate.

The analysis-of-variance method can, therefore, be recommended for general use because

(a) it gives high efficiency for values of σ_p^2/σ_w^2 most likely to occur in practice,
(b) it is most suitable for routine application on an electronic computer using the method of Section 2.4.3.

2.5. *Estimation of Mean Rotation Effects and Regressions: Unweighted Analysis*

The estimation of mean rotation effects over the years and the regressions of $\bar{y}_{ij}$ on a set of seasonal and trend variates x_{im} for a single years $\times$ tests table can now be considered. It is convenient to use matrix notation. Frequent use is made of matrices $\mathbf{1}$, consisting of a single column of unit elements. Where necessary a suffix will be used to denote the order of $\mathbf{1}$ or $\mathbf{I}$, an identity matrix. Thus $\mathbf{1}_N$ is of order $N\times 1$; $\mathbf{I}_N$ is of order $N\times N$. Kronecker products of two matrices are also used. These will be denoted by $\times$. Thus

$$\mathbf{A}\times\mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & a_{13}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & a_{23}\mathbf{B} & \dots \\ a_{31}\mathbf{B} & a_{32}\mathbf{B} & a_{33}\mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{2.5.1}$$

With this notation equation (2.1.1) can be rewritten in the following form:

$$\mathbf{y} = \mathbf{J}_t\boldsymbol{\tau}+\mathbf{J}_b\boldsymbol{\beta}+\mathbf{u}, \tag{2.5.2}$$

where $\boldsymbol{\tau}$ is the $t\times 1$ matrix of the τ_j, $\boldsymbol{\beta}$ is the $qt\times 1$ matrix of the β_{jm} and $\mathbf{u}$ is an $Nt\times 1$ matrix of appropriate terms in ψ_{ij} and ϵ_{ij}. The matrix

$$\mathbf{y}' = (\mathbf{y}_1'\,\mathbf{y}_2'\,\mathbf{y}_3'\ldots\mathbf{y}_t') \tag{2.5.3}$$

is of order $1\times Nt$ where $\mathbf{y}_j$ is the $N\times 1$ matrix of the mean yields $\bar{y}_{ij}$ for test j. Also

$$\mathbf{J}_t = \mathbf{I}_t\times\mathbf{1}_N \tag{2.5.4}$$

and

$$\mathbf{J}_b = \mathbf{I}_t\times\mathbf{x}, \tag{2.5.5}$$

where $\mathbf{x}$ is the $N\times q$ matrix of x'_{im}. It should be noted that

$$\mathbf{1}_N'\,\mathbf{x} = 0 \tag{2.5.6}$$

with the definition of x'_{im} of Section 2.1.

The simplest method for estimating $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ is to minimize

$$(\mathbf{y}'-\mathbf{t}'\,\mathbf{J}_t'-\mathbf{b}'\,\mathbf{J}_b')(\mathbf{y}-\mathbf{J}_t\mathbf{t}-\mathbf{J}_b\mathbf{b}), \tag{2.5.7}$$

for variations in $\mathbf{t}$ and $\mathbf{b}$, without regard to the correlations between elements of $\mathbf{u}$ due to recurrence of yield values on the same plots. This method will be referred to as the unweighted analysis to distinguish it from methods in which different weights are given to contrasts between plot totals and "within-plot" contrasts, i.e. contrasts that are orthogonal with plot differences.

Methods for determining weighted estimates will be discussed in Sections 2.6 and 2.7. The present Section deals with the unweighted analysis of a single years × tests table constructed as described in Section 2.1 for a replicated experiment of basic or reduced design.

2.5.1. *Estimates of means and regressions*

The unweighted estimates, $\mathbf{t}$ and $\mathbf{b}$, of $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ respectively are simply

$$\mathbf{t} = (\mathbf{I}_t\times N^{-1}\,\mathbf{1}_N')\,\mathbf{y} \tag{2.5.8}$$

and

$$\mathbf{b} = \{\mathbf{I}_t\times(\mathbf{x}'\mathbf{x})^{-1}\,\mathbf{x}'\}\,\mathbf{y}, \tag{2.5.9}$$

i.e. the ordinary estimates of means and regressions for each column of the years × tests table.

Plot correlations and heterogeneity in the elements of $\mathbf{u}$ must be allowed for in deriving expressions for the variances of $\mathbf{t}$ and $\mathbf{b}$ by including suitable multiples of the components σ_p^2 and α_{jh} defined in Section 2.2. The coefficients of σ_p^2 are determined partly by the coefficients of the $\bar{y}_{ij}$ in equations (2.5.8) and (2.5.9) and partly by $N\times N$ matrices of the type $\mathbf{H}_j\mathbf{H}_j', \mathbf{H}_j\mathbf{H}_h'$ where $\mathbf{H}_j$ is an $N\times p$ matrix related to the years × plots incidence matrix $\mathbf{G}$ defined in Section 2.4.1. If test j is on plots k in year i then the kth element in row i of $\mathbf{H}_j$ is equal to 1 and all other elements in row i are zero. We define $\mathbf{H}$ as $\Sigma_j\,\mathbf{H}_j$. For any basic design or a reduced design when the analysis includes all the tests of the test crop, $\mathbf{H}$ is equal to $\mathbf{G}$. If there are two or more groups of tests $\mathbf{G}$ is given by the sum of $\mathbf{H}$ over all groups. The somewhat clumsy matrices $\mathbf{H}_j$ need not be used in practice as $\mathbf{H}_j\mathbf{H}_j'$ and $\mathbf{H}_j\mathbf{H}_h'$ can be defined directly.

If the rotation with test j has a period of c_j years, $\mathbf{H}_j\mathbf{H}_j'$ is the $N \times N$ matrix such that

(i) all principal diagonal elements are 1;
(ii) the element 1 occurs in every c_jth position to the left and right of the diagonal in every row;
(iii) all other elements are zero.

For example, when $N = 6$ and $c_j = 4$, $\mathbf{H}_j\mathbf{H}_j'$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrices $\mathbf{H}_j\mathbf{H}_h'$, with $j \neq h$, are non-zero only when tests j and h are in different phases of the same rotation. If test j is on plots k in year i and test h is on the same plots in years $i', i'', \ldots$, the elements $(i, i'), (i, i''), \ldots$ of $\mathbf{H}_j\mathbf{H}_h'$ are unit elements and all other elements of row i are zero. For example, $\mathbf{H}_2\mathbf{H}_3'$ for any six consecutive years of the design of Table 4 is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

where $j = 2$, $h = 3$ refer to R_2 and R_3 respectively.

For a design of the type discussed in Section 1.3.1

$$\left.\begin{aligned} t\mathbf{H}_j\mathbf{H}_j' &= \mathbf{H}\mathbf{H}' \quad \text{for all } j, \\ \mathbf{H}_j\mathbf{H}_h' &= 0 \qquad \text{for } j \neq h. \end{aligned}\right\} \tag{2.5.10}$$

If we now define the $p \times Nt$ matrix

$$\mathbf{J}_p' = (\mathbf{H}_1'\mathbf{H}_2'\mathbf{H}_3' \ldots \mathbf{H}_t') \tag{2.5.11}$$

the variance matrix of the elements of $\mathbf{y}$ for model I of Section 2.1 can be written

$$\text{var}(\mathbf{y}) = \mathbf{V} = \frac{\mathbf{J}_p\mathbf{J}_p'\sigma_p^2}{r} + \mathbf{C}_w^{-1} \times \mathbf{I}_N, \tag{2.5.12}$$

where

$$\mathbf{C}_w^{-1} = \boldsymbol{\alpha} + \frac{\mathbf{I}_t\sigma_w^2}{r} \tag{2.5.13}$$

and $\boldsymbol{\alpha}$ is the $t \times t$ matrix of the α_{jh}.

The variances and covariances of the elements t_j of $\mathbf{t}$ are therefore given by

$$\operatorname{cov}(t_j, t_h) = \frac{\mathbf{1}'_N \mathbf{H}_j \mathbf{H}'_h \mathbf{1}_N}{N^2} \frac{\sigma_p^2}{r} + \frac{r\alpha_{jh} + \delta\sigma_w^2}{Nr}, \tag{2.5.14}$$

where

$$\delta = 0 \text{ when } j \neq h$$

and

$$\delta = 1 \text{ when } j = h. \tag{2.5.15}$$

The values of $\mathbf{1}'\mathbf{H}_j \mathbf{H}'_j \mathbf{1}$ and $\mathbf{1}'\mathbf{H}_j \mathbf{H}'_h \mathbf{1}$ are simply the numbers of unit elements in the matrices $\mathbf{H}_j \mathbf{H}'_j$ and $\mathbf{H}_j \mathbf{H}'_h$ respectively.

Similarly

$$\operatorname{cov}(\mathbf{b}_j, \mathbf{b}_h) = (\mathbf{x}'\mathbf{x})^{-1} \mathbf{x}' \mathbf{H}_j \mathbf{H}'_h \mathbf{x} (\mathbf{x}'\mathbf{x})^{-1} \frac{\sigma_p^2}{r} + (\mathbf{x}'\mathbf{x})^{-1} \left(\alpha_{jh} + \frac{\delta\sigma_w^2}{r} \right), \tag{2.5.16}$$

where the elements of $\mathbf{b}_j$ are the q regressions for test j, and

$$\operatorname{cov}(t_j, \mathbf{b}_h) = \frac{\mathbf{1}'\mathbf{H}_j \mathbf{H}'_h \mathbf{x}(\mathbf{x}'\mathbf{x})^{-1} \sigma_p^2}{Nr}. \tag{2.5.17}$$

When N is a multiple of c_j, i.e. there are no incomplete cycles of the rotation with test j, the sums of the elements in the columns of $\mathbf{H}_j \mathbf{H}'_h$ are equal so that $\operatorname{cov}(t_j, \mathbf{b}_h)$ is zero for all h.

The α_{jh} are often large because they include a component of the seasonal variation in mean level of yield. This is of no consequence as only comparisons between the regressions for different tests are of interest. The seasonal component is eliminated from the variances of these comparisons. In practice it may be convenient to replace the α_{jh} by the α'_{jh} of equation (2.2.3).

Estimates of σ_p^2 and σ_w^2 are already available (Sections 2.3 and 2.4). There remain the problems of (a) testing whether there are any real variations in rotation effects from year to year, and (b) if so, estimating the α_{jh}. A suitable method for (a) is to compare the residual years × tests sum of squares of the analysis of variance with an estimate of its expected value when

$$\sum_j \alpha_{jj} - \frac{\sum_{j,h} \alpha_{jh}}{t} = 0, \quad \text{i.e. } \sigma_{ty}^2 = 0. \tag{2.5.18}$$

2.5.2. *Analysis of variance*

Table 10 shows the analysis of variance of the years × tests table in units of a single value of the table. The Y_j are given by

$$Y_j = \sum_i \bar{y}_{ij}.$$

On the null hypothesis (2.5.18) the expected value of S_{ty} is

$$\frac{t(N-q-1)(t-1)\sigma_w^2 + (t \sum_j \operatorname{tr} \mathbf{H}_j \mathbf{H}'_j \mathbf{F} - \operatorname{tr} \mathbf{H}\mathbf{H}'\mathbf{F})\sigma_p^2}{rt}, \tag{2.5.19}$$

where

$$\mathbf{F} = \mathbf{I}_N - \frac{\mathbf{1}\mathbf{1}'}{N} - \mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'. \tag{2.5.20}$$

If S_{ty0} is obtained by substituting estimates of σ_p^2 and σ_w^2 in (2.5.19) the ratio

$$\frac{(N-q-1)(t-1)\,S_{ty}}{S_{ty0}} \qquad (2.5.21)$$

can be tested in the χ^2 distribution with $(N-q-1)(t-1)$ degrees of freedom provided, as will usually be the case, that σ_p^2 and σ_w^2 are estimated from a large number of error degrees of freedom.

An estimate of σ_{ty}^2 is given by

$$\hat{\sigma}_{ty}^2 = \frac{S_{ty}-S_{ty0}}{(N-q-1)(t-1)}. \qquad (2.5.22)$$

If σ_{ty}^2 is homogeneous $\hat{\sigma}_{ty}^2$ can be substituted for α_{jj} in equations (2.5.14), (2.5.16) and (2.5.17) and α_{jh} replaced by zero when j is not equal to h. In general, however, separate estimates of σ_{tys}^2 derived from estimates of α_{jh} are likely to be required for different contrast between the tests.

TABLE 10

Analysis of variance

	d.f.	S.S.
Years	$N-1$	
Rotation effects (tests)	$t-1$	$S_t = \sum_j t_j\, Y_j - \frac{\sum_j t_j \sum_j Y_j}{t}$
Regressions	$q(t-1)$	$S_b = \sum_j \mathbf{b}_j' \mathbf{x}'\mathbf{y}_j - \frac{\sum_j \mathbf{b}_j' \sum_j \mathbf{x}'\mathbf{y}_j}{t}$
Residual years × tests	$(N-q-1)(t-1)$	S_{tv}
Total	$Nt-1$	

As an example of the use of the matrices $\mathbf{H}_j$, $\mathbf{H}$ and $\mathbf{F}$ in equation (2.5.19) the analysis of 5 years of the design of Table 4 will be considered. Here $N = 5$ and we will take $q = 1$, $t = 4$, $r = 3$ and $x' = (-2, -1, 0, 1, 2)$. The $\mathbf{H}_j\mathbf{H}_j'$ are defined by $c_1 = 3$, $c_2 = c_3 = c_4 = 4$. Hence

$$\mathbf{HH}' = \begin{bmatrix} 4 & 1 & 0 & 2 & 3 \\ 1 & 4 & 1 & 0 & 2 \\ 0 & 1 & 4 & 1 & 0 \\ 2 & 0 & 1 & 4 & 1 \\ 3 & 2 & 0 & 1 & 4 \end{bmatrix}, \qquad \mathbf{F} = \frac{1}{10}\begin{bmatrix} 4 & -4 & -2 & 0 & 2 \\ -4 & 7 & -2 & -1 & 0 \\ -2 & -2 & 8 & -2 & -2 \\ 0 & -1 & -2 & 7 & -4 \\ 2 & 0 & -2 & -4 & 4 \end{bmatrix},$$

$\operatorname{tr}\mathbf{H}_j\mathbf{H}_j'\mathbf{F} = 3$ when $j = 1$, and 3·4 when $j = 2$, 3, or 4, $\operatorname{tr}\mathbf{HH}'\mathbf{F} = 10{\cdot}8$ and

$$\mathscr{E}(S_{ty0}) = 3\sigma_w^2 + 3{\cdot}5\sigma_p^2.$$

2.5.3. *Estimates of* α_{jh} *and* σ^2_{tys}

Estimates, $\hat{\alpha}_{jh}$ say, of the α_{jh} can be obtained from the following residual sums of squares and products:

$$\left.\begin{aligned} S_{jh} &= (\mathbf{y}'_j - t_j \mathbf{1}' - \mathbf{b}'_j \mathbf{x}')(\mathbf{y}_h - \mathbf{1} t_h - \mathbf{x}\mathbf{b}_h) \\ &= \mathbf{y}'_j \mathbf{y}_h - \frac{\mathbf{1}'\mathbf{y}_j \mathbf{y}'_h \mathbf{1}}{N} - \mathbf{y}'_j \mathbf{x}(\mathbf{x}'\mathbf{x})^{-1}\mathbf{x}'\mathbf{y}_h. \end{aligned}\right\} \tag{2.5.23}$$

If S_{jh0} is the estimate of

$$\frac{(N-q-1)\,\delta\sigma^2_w + \operatorname{tr}\mathbf{H}_h\mathbf{H}'_j\mathbf{F}\sigma^2_p}{r} \tag{2.5.24}$$

then

$$\hat{\alpha}_{jh} = \frac{S_{jh} - S_{jh0}}{(N-q-1)}.$$

These estimates are not fully efficient but are considered to be adequate for most practical purposes. Estimates of σ^2_{tys} can be obtained by substituting $\hat{\alpha}_{jh}$ for α_{jh} in equation (2.2.5).

2.5.4. *Analysis on model II*

The analysis for equation (2.1.2) (model II) is very similar. The estimates of $\bar{\tau}_j$ and β_{jm} are given by equations (2.5.8) and (2.5.9), as before, but equations (2.5.12) and (2.5.14) require modification. Values of $\mathbf{V}_{\mathrm{I}}$ and $\mathbf{V}_{\mathrm{II}}$, the variance matrices of $\mathbf{y}$ on models I and II, respectively, are connected by the following relationship:

$$\mathbf{V}_{\mathrm{II}} = \mathbf{V}_{\mathrm{I}} - N^{-1}\boldsymbol{\alpha} \times \mathbf{1}_N \mathbf{1}'_N. \tag{2.5.25}$$

It follows that the term in α_{jh} must be deleted from equation (2.5.14) and that equations (2.5.16) and (2.5.17) are unaltered.

These results can be made more general. Estimates of $\bar{\boldsymbol{\tau}}$ (or $\boldsymbol{\tau}$) and $\boldsymbol{\beta}$ can be expressed in the forms $\mathbf{W}_t\mathbf{y}$ and $\mathbf{W}_b\mathbf{y}$, respectively, where $\mathbf{W}_t$ is a $t \times Nt$ matrix such that

$$\mathbf{W}_t(\mathbf{I}_t \times \mathbf{1}_N) = \mathbf{I}_t \tag{2.5.26}$$

and $\mathbf{W}_b$ is a $tq \times Nt$ matrix such that

$$\mathbf{W}_b(\mathbf{I}_t \times \mathbf{1}_N) = 0. \tag{2.5.27}$$

Hence

$$\operatorname{var}_{\mathrm{II}}(\mathbf{W}_t\mathbf{y}) = \operatorname{var}_{\mathrm{I}}(\mathbf{W}_t\mathbf{y}) - N^{-1}\boldsymbol{\alpha}, \tag{2.5.28}$$

$$\operatorname{var}_{\mathrm{II}}(\mathbf{W}_b\mathbf{y}) = \operatorname{var}_{\mathrm{I}}(\mathbf{W}_b\mathbf{y}), \tag{2.5.29}$$

and

$$\operatorname{cov}_{\mathrm{II}}(\mathbf{W}_t\mathbf{y}, \mathbf{W}_b\mathbf{y}) = \operatorname{cov}_{\mathrm{I}}(\mathbf{W}_t\mathbf{y}, \mathbf{W}_b\mathbf{y}), \tag{2.5.30}$$

where $\operatorname{var}_{\mathrm{I}}$, $\operatorname{var}_{\mathrm{II}}$, $\operatorname{cov}_{\mathrm{I}}$ and $\operatorname{cov}_{\mathrm{II}}$ denote variances and covariances on models I and II.

2.5.5. *Conditions for unweighted means to be efficient*

The unweighted mean

$$N^{-1}\sum_i \bar{y}_{ij}$$

is an efficient estimate of τ_j or $\bar{\tau}_j$ on the two-component model for error with $\sigma^2_p \neq 0$ when the cycles of the rotation that includes test j are complete, i.e. N is a multiple of the length of the rotation. This result can be demonstrated readily by considering the covariances

$$\operatorname{cov}\left(\bar{y}_{ih}, N^{-1}\sum_i \bar{y}_{ij}\right); \tag{2.5.31}$$

these can be determined from $\mathbf{V}$ (equation (2.5.12)). The unweighted mean is efficient if the covariances for all i and h are independent of i. For any test h in a different rotation to test j the covariances involve terms in α_{jh} only and are therefore independent of i. When test h is in the same rotation as test j the covariances also involve terms of σ_p^2 (and σ_w^2 if $j = h$); they are equal for all i if N is a multiple of the length of the rotation.

It follows that the unweighted means for all tests are efficient when the cycles of the whole experiment are complete, i.e. when N is a common multiple of the periods of the individual rotations. The unweighted means are also efficient for small values of N such that no two annual test means are derived from the same plots. In other cases some or all of the unweighted means are not efficient but the loss of information is usually very small.

2.6. *Weighted Analysis*

Unweighted regressions, on the other hand, are inefficient whether the cycles are complete or not and, as will be shown, losses of efficiency can be serious in some circumstances. Cochran (1939) has described a method for increasing the accuracy of estimated polynomial regression coefficients in fixed-rotation experiments by assigning different weights to between-plot and within-plot contrasts. Similar methods can be used for experiments on different crop rotations.

The analysis for a single years × tests table in the designs of Section 1.3.1 will be considered first. It will be supposed that the N years consist of n complete cycles ($n > 1$) and that there are c series. Initially σ_{ty}^2 will be assumed to be homogeneous, as defined in Section 2.2.

2.6.1. *Weighted analysis: designs of Section* 1.3.1 *with* σ_{ty}^2 *homogeneous*

Suppose that the $N \times t$ years × tests table is divided into c separate $n \times t$ tables, one for each series, and that the following quantities are calculated:

$\mathbf{Z}_{wj}, \mathbf{Z}_{pj}$: two $q \times 1$ matrices, for each test, of the sums of products of $\bar{y}_{ij}$ with the x_{im} within series and between series respectively.

$\mathbf{X}_w, \mathbf{X}_p$: two $q \times q$ matrices of the sums of squares and products of the x_{im} within series and between series respectively.

The required weighted coefficients for tests j are given by the elements of $\hat{\mathbf{b}}_j$, where

$$\hat{\mathbf{b}}_j = (v_p^{-1}\mathbf{X}_p + v_w^{-1}\mathbf{X}_w)^{-1}(v_p^{-1}\mathbf{Z}_{pj} + v_w^{-1}\mathbf{Z}_{wj}), \tag{2.6.1}$$

and

$$v_p = \sigma_{ty}^2 + \frac{n\sigma_p^2 + \sigma_w^2}{r}, \tag{2.6.2}$$

$$v_w = \sigma_{ty}^2 + \frac{\sigma_w^2}{r}. \tag{2.6.3}$$

These regressions are efficient (apart from losses in estimating v_p and v_w) if σ_{ty}^2 is homogeneous. Their variance matrix is

$$(v_p^{-1}\mathbf{X}_p + v_w^{-1}\mathbf{X}_w)^{-1}. \tag{2.6.4}$$

Within-series estimates of the regressions for any $q \leqslant N-c$, and their variances, are obtained by putting $v_p^{-1} = 0$ in equations (2.6.1) and (2.6.4). It will be shown in Section 2.6.3 that these estimates are preferable to the unweighted estimates and almost as efficient as the weighted estimates in some circumstances. Provided that $q \leqslant c-2$ separate between-series estimates can also be obtained by substituting $v_w^{-1} = 0$ in equation (2.6.1) but these estimates can be very inaccurate. The weighted estimates are available for any $q \leqslant N-1$ but if $q = N-1$ they are, of course, the same as the unweighted estimates.

In practice the weights v_p^{-1} and v_w^{-1} must be estimated. The methods of Sections 2.3 and 2.5.2 are appropriate. A somewhat simpler but less accurate method of estimating σ_{ty}^2 will also be considered. This method is based on the between-series and within-series analysis of variance shown in Table 11 in units of a single mean yield $\bar{y}_{ij}$.

TABLE 11

Analysis of variance

	d.f.	M.S.	$\mathscr{E}$(M.S.)
Between series			
Series (years)	$c-1$		
Tests	$t-1$		
Regressions	$q(t-1)$		
Residual years × tests	$(c-q-1)(t-1)$	M_p	$(r\sigma_{ty}^2+n\sigma_p^2+\sigma_w^2)/r$
Within series			
Years	$c(n-1)$		
Regressions	$q(t-1)$		
Residual years × tests	$(N-c-q)(t-1)$	M_w	$(r\sigma_{ty}^2+\sigma_w^2)/r$
	$Nt-1$		

The tests sum of squares is obtained as in Table 10. The between-series regressions sum of squares is given by

$$\sum_j \mathbf{Z}'_{pj}\mathbf{X}_p^{-1}\mathbf{Z}_{pj} - \frac{\left(\sum_j \mathbf{Z}'_{pj}\right)\mathbf{X}_p^{-1}\left(\sum_j \mathbf{Z}_{pj}\right)}{t} \tag{2.6.5}$$

and the within-series regressions sum of squares by a similar expression with the suffix p replaced by w.

The required estimate of σ_{ty}^2 is obtained by pooling the two residual years × tests sums of squares and equating the pooled sum of squares to its expected value using the estimates of σ_p^2 and σ_w^2 of Section 2.3.

The advantage of this method of estimating the weights rests in its use of a standard analysis-of-variance technique and simple expressions for $\mathscr{E}(M_p)$ and $\mathscr{E}(M_w)$. Its loss of accuracy results from the loss of $q(t-1)$ degrees of freedom for residual years × tests representing the differences between the within-series regressions and the between-series regressions. The method is not available if $q > c-2$.

The even simpler method of estimating the weights directly from M_p and M_w can be very inaccurate and is not recommended for replicated designs.

2.6.2. *Weighted analysis: σ_{ty}^2 not homogeneous*

Heterogeneity in the years × tests interaction results in loss of efficiency in the method of Section 2.6.1. Improved estimates of regressions for particular contrasts can be obtained by replacing the estimate of σ_{ty}^2 by estimates of the appropriate σ_{tys}^2 (Section 2.5.3), but the results will in general be inconsistent in that, for example,

$$b_{1,2}+b_{2,3} \neq b_{1,3}$$

where $b_{j,j'}$ is the estimated regression of $\bar{y}_{ij}-\bar{y}_{ij'}$ on one of the x_{im}.

The following method gives consistent results and tends to be fully efficient as the errors in the estimates of α_{jh}, σ_p^2 and σ_w^2 tend to zero. It uses weights determined from the estimates of the $t \times t$ matrices $\mathbf{C}_w^{-1}$ defined by equation (2.5.13) and $\mathbf{C}_p^{-1}$ given by

$$\mathbf{C}_p^{-1} = \mathbf{C}_w^{-1} + \frac{n\mathbf{I}_t \sigma_p^2}{r}. \qquad (2.6.6)$$

Matrices $\mathbf{Z}_w$ and $\mathbf{Z}_p$ are defined as follows:

$$\mathbf{Z}_w' = (\mathbf{Z}_{w1}', \mathbf{Z}_{w2}', \mathbf{Z}_{w3}', \ldots, \mathbf{Z}_{wt}'), \qquad (2.6.7)$$

$$\mathbf{Z}_p' = (\mathbf{Z}_{p1}', \mathbf{Z}_{p2}', \mathbf{Z}_{p3}', \ldots, \mathbf{Z}_{pt}'), \qquad (2.6.8)$$

where $\mathbf{Z}_{w1}$, etc. are as in Section 2.6.1. The efficient estimates of regressions (ignoring errors in the weights) are given by the elements of $\hat{\mathbf{b}}$ where

$$\hat{\mathbf{b}} = (\mathbf{C}_w \times \mathbf{X}_w + \mathbf{C}_p \times \mathbf{X}_p)^{-1}\{(\mathbf{C}_w \times \mathbf{I}_q)\mathbf{Z}_w + (\mathbf{C}_p \times \mathbf{I}_q)\mathbf{Z}_p\}. \qquad (2.6.9)$$

The variance matrix of the regressions is

$$\operatorname{var}(\hat{\mathbf{b}}) = (\mathbf{C}_w \times \mathbf{X}_w + \mathbf{C}_p \times \mathbf{X}_p)^{-1}. \qquad (2.6.10)$$

The above analysis involves contrasts between different years as well as contrasts within years. The between-year contrasts are given appropriately small weight when year-to-year variations in mean yield are large. In practice it may be desirable to confine the analysis to within-year contrasts, particularly when the number of years is small. This can be done by replacing the α_{jh} in the expressions for $\mathbf{C}_w$ and $\mathbf{C}_p$ by the α_{jh}' of equation (2.2.3), multiplying the modified matrices by $\mathbf{I}_t - t^{-1}\mathbf{1}_t\mathbf{1}_t'$ and introducing restrictions $(\mathbf{1}_t' \times \mathbf{I}_q)\hat{\mathbf{b}} = 0$.

2.6.3. *Efficiencies of unweighted regressions and within-series regressions*

Theoretical expressions, ignoring errors in the weights, are readily available for the efficiencies of unweighted regressions b_j and within-series regressions b_{wj} relative to weighted regressions $\hat{b}_j$ under the conditions of the analysis of Section 2.6.1. These expressions provide useful guidance on the choice of one of the three types of estimate. Only a single variate x_i' is considered but the results can be applied to multiple regression.

The following quantities are required:

$$\lambda = \frac{c(n-1)}{c-1}, \qquad (2.6.11)$$

$$\eta = \frac{X_w}{\lambda X_p} \quad (0 < \eta < \infty), \qquad (2.6.12)$$

where X_w and X_p are the within-series and between-series sums of squares of the x_i',

$$\theta = \frac{\sigma_p^2}{r\sigma_{ty}^2 + \sigma_w^2} \tag{2.6.13}$$

and

$$\kappa = n\theta + 1 \quad (\kappa \geqslant 1). \tag{2.6.14}$$

The efficiencies of b_{wj} and b_j are then

$$E_w = \frac{\kappa\lambda\eta}{1+\kappa\lambda\eta} \tag{2.6.15}$$

and

$$E = \frac{\kappa(\lambda\eta+1)^2}{(1+\kappa\lambda\eta)(\lambda\eta+\kappa)}, \tag{2.6.16}$$

respectively.

The ratio $\theta/(1+\theta)$ is generally smaller than the correlation between the errors of yield values recurring on the same plots (Section 2.2). Hence, practical values of θ are likely to lie in the range $0 < \theta < 1$. The value of η depends on the intraclass correlation, ρ' say, of the x_i' where the series define the classes:

$$\eta = \frac{1-\rho'}{1+\rho'(n-1)}. \tag{2.6.17}$$

Thus η lies in the range $0 \leqslant \eta < 1$ when ρ' is positive, is equal to 1 when $\rho' = 0$ and greater than 1 when ρ' is negative. An average value of η of 1 can therefore be used when considering regressions on seasonal variates from which trends have been eliminated. For linear and quadratic regression, using the orthogonal polynomial coefficients ξ' tabulated by Fisher and Yates (1963), η is greater than 1. For the linear coefficients

$$\eta = \frac{c(n+1)}{c+1}; \tag{2.6.18}$$

values for the quadratic coefficients are even larger. When higher order polynomials are used η can be less than 1, but these cases do not appear to occur very often in practice.

The unweighted estimate b_j is efficient when $\kappa = 1$, i.e. $\theta = 0$, but its efficiency falls off as κ increases. When $\kappa < 2$, $E > E_w$ for all $\lambda\eta$. When $\kappa > 2$ the within-series estimate b_{wj} is more efficient than b_j if, and only if,

$$\kappa > \frac{1+2\lambda\eta}{\lambda\eta}. \tag{2.6.19}$$

For very large κ the estimate b_{wj} tends to be efficient. Table 12 gives the efficiencies of b_j and b_{wj} for various values of n, c and θ when $\eta = 1$. When $\eta > 1$ the efficiencies of both b_j and b_{wj} are greater than those shown. When $\eta < 1$ the efficiencies are smaller for b_{wj} and, provided that $\lambda^2\eta > 1$, for b_j also. The efficiency of b_j takes a minimum value of

$$\frac{4\kappa}{(\kappa+1)^2} \tag{2.6.20}$$

when $\lambda\eta = 1$.

Table 12 shows that neither of the two estimates, b_j and b_{wj}, adequately covers the range of values of n, c and θ that may be met in practice. For any particular set of values of n, c and θ, however, one of the two estimates is likely to be reasonably efficient. The efficiencies resulting from the following two procedures will be considered:

(1) Choose the more efficient of b_j and b_{wj}.
(2) If $\kappa < 2$ choose b_j and if $\kappa \geqslant 2$ choose b_{wj}.

Procedure (1) cannot be used exactly in multiple regression on non-orthogonal variates since one type of estimate must be chosen for all variates. Procedure (2) results in the within-series estimates being used on some occasions when the unweighted estimates b_j are more efficient. The efficiency of procedure (1) is the value of E_w or E, whichever is greater. This will be denoted by E'. For any given $\lambda\eta$ the minimum value of E' for variations in κ is

$$E'_{\min} = \frac{1+2\lambda\eta}{2+2\lambda\eta}. \tag{2.6.21}$$

The efficiency of procedure (2) is E if $\kappa < 2$ and E_w if $\kappa \geqslant 2$. It will be denoted by E''. The minimum value of E'' is

$$E''_{\min} = \frac{2\lambda\eta}{1+2\lambda\eta}. \tag{2.6.22}$$

Table 13 shows values of $E'_{\min}$ and $E''_{\min}$ for various n and c when the x'_i represent (i) a seasonal variate, with $\eta = 1$, and (ii) the linear, (iii) quadratic, (iv) cubic orthogonal polynomial function values tabulated by Fisher and Yates (1963). Except when $n = 2$ the efficiencies are reasonably high, suggesting that b_j and b_{wj} may be adequate for many practical purposes provided that a suitable choice is made between them.

TABLE 12

Percentage efficiencies of (i) an unweighted regression coefficient and (ii) a within-series regression coefficient when $\eta = 1$

n	θ	$c = 3$ (i)	$c = 3$ (ii)	$c = 6$ (i)	$c = 6$ (ii)
2	0	100	60	100	55
	0·5	89	75	89	71
	1	76	82	75	78
4	0	100	82	100	78
	0·5	83	93	82	92
	1	68	96	65	95
6	0	100	88	100	86
	0·5	81	97	78	96
	1	65	98	61	98

In practice, errors in the estimates of σ^2_{ty}, σ^2_p and σ^2_w contribute to the errors of the $\hat{b}_j$. They also result in an incorrect choice being made between b_j and b_{wj} in some circumstances. In consequence the actual efficiencies of b_{wj} and b_j are somewhat larger than those given by equations (2.6.15) and (2.6.16) or shown in Table 12. Also, practical applications of procedures (1) or (2) described above may, in special cases, result in smaller efficiencies than the minimum efficiencies given by equations (2.6.21) and (2.6.22) or shown in Table 13.

In general, the unweighted analysis (Section 2.5) appears to be the most useful in practice. The simple weighted estimate $(b_j + b_{wj})/2$ can be used if $\kappa > 1{\cdot}5$ and b_{wj} if $\kappa > 5$; the fully weighted analysis is unlikely to be required in practice.

TABLE 13

Percentage values of $E'_{\min}$ and $E''_{\min}$ for regressions on (i) seasonal variates with $\eta = 1$, (ii) linear, (iii) quadratic, (iv) cubic orthogonal polynomial coefficients

c	n	$E'_{\min}$				$E''_{\min}$			
		(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)
3	2	80	89	98	78	75	87	98	71
	4	91	97	100	94	90	97	100	93
	6	94	99	100	97	94	99	100	97
6	2	77	88	97	82	71	86	97	78
	4	89	97	100	94	88	97	100	93
	6	93	99	100	97	92	99	100	97

2.7. *Weighted Analysis: General Method for Reduced or Basic Designs*

In this section the analysis described in Section 2.6.2 is extended to cover any number of years for an experiment of reduced or basic design. The within-plot analysis, which may be useful when residual years × tests variations are small, will be obtained as a special case of the general method. The weighted estimates, $\hat{\mathbf{t}}$ and $\hat{\mathbf{b}}$, are obtained by minimizing

$$(\mathbf{y}' - \hat{\mathbf{t}}'\mathbf{J}'_t - \hat{\mathbf{b}}'\mathbf{J}'_b)\,\mathbf{V}^{-1}(\mathbf{y} - \mathbf{J}_t\hat{\mathbf{t}} - \mathbf{J}_b\hat{\mathbf{b}}), \tag{2.7.1}$$

with matrices as defined in Section 2.5. The matrix $\mathbf{V}^{-1}$ can be expressed conveniently in the following form:

$$\mathbf{V}^{-1} = (\mathbf{C}_w \times \mathbf{I}_N) - \frac{\sigma_p^2}{r}(\mathbf{C}_w \times \mathbf{I}_N)\,\mathbf{J}_p\,\mathbf{V}_p^{-1}\,\mathbf{J}'_p(\mathbf{C}_w \times \mathbf{I}_N), \tag{2.7.2}$$

where

$$\mathbf{V}_p = \mathbf{I}_p + \frac{\sigma_p^2}{r}\,\mathbf{J}'_p(\mathbf{C}_w \times \mathbf{I}_N)\,\mathbf{J}_p. \tag{2.7.3}$$

Use of equation (2.7.2) avoids direct inversion of $\mathbf{V}$, which is often very large.

The following notation will be used for certain matrices of coefficients and numerical quantities arising in the normal equations:

$$\left.\begin{aligned} \mathbf{J}'_t(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_p &= \mathbf{K}_t,\\ \mathbf{J}'_b(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_p &= \mathbf{K}_b,\\ \mathbf{J}'_p(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_p &= \mathbf{K}_p, \end{aligned}\right\} \tag{2.7.4}$$

and

$$\left.\begin{aligned} \mathbf{J}'_t(\mathbf{C}_w \times \mathbf{I})\,\mathbf{y} &= \mathbf{T},\\ \mathbf{J}'_b(\mathbf{C}_w \times \mathbf{I})\,\mathbf{y} &= \mathbf{Z},\\ \mathbf{J}'_p(\mathbf{C}_w \times \mathbf{I})\,\mathbf{y} &= \mathbf{P}. \end{aligned}\right\} \tag{2.7.5}$$

Methods for evaluating these matrices are discussed below. The remaining matrices of coefficients required in the normal equations simplify immediately. We have

$$\left.\begin{aligned} \mathbf{J}_t'(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_t &= N\mathbf{C}_w, \\ \mathbf{J}_t'(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_b &= 0 \quad \left(\text{as } \sum_i x_{im} = 0\right), \\ \mathbf{J}_b'(\mathbf{C}_w \times \mathbf{I})\,\mathbf{J}_b &= \mathbf{C}_w \times \mathbf{x}'\mathbf{x}. \end{aligned}\right\} \tag{2.7.6}$$

The matrices of equations (2.7.4) and (2.7.5) can be determined without using $\mathbf{J}_t$, $\mathbf{J}_b$ and $\mathbf{J}_p$ directly. We have

$$\mathbf{T} = (\mathbf{C}_w \times \mathbf{1}_N')\,\mathbf{y} \tag{2.7.7}$$

and

$$\mathbf{Z} = (\mathbf{C}_w \times \mathbf{x}')\,\mathbf{y}. \tag{2.7.8}$$

The matrix $\mathbf{P}$ can be obtained by forming a tests $\times$ plots table of linear functions of the $\bar{y}_{ij}$. If plots k have tests $j', j'', \ldots$ in sets of years $1, 2, \ldots$ respectively, element (j, k) of the table is given by

$$c_{jj'}\, S_1(\bar{y}_{ij}) + c_{jj''}\, S_2(\bar{y}_{ij}) + \ldots, \tag{2.7.9}$$

where $S_1, S_2, \ldots$ denote summation over sets of years $1, 2, \ldots$ respectively and c_{jh} is element (j, h) of $\mathbf{C}_w$. The elements of the $p \times 1$ matrix $\mathbf{P}$ are the sums of the columns of the tests $\times$ plots table.

Element (j, k) of $\mathbf{K}_t$ is given by (2.7.9) with each $\bar{y}_{ij}$ replaced by 1; $\mathbf{K}_b'$ is given by $(\mathbf{K}_{b1}', \mathbf{K}_{b2}', \ldots, \mathbf{K}_{bt}')$, where element (m, k) of the $q \times p$ matrix $\mathbf{K}_{bj}$ is given by (2.7.9) with the $\bar{y}_{ij}$ replaced by x_{im}. The $p \times p$ matrix $\mathbf{K}_p$ is given by

$$\mathbf{K}_p = \sum_{j,h} \mathbf{H}_j' \mathbf{H}_h\, c_{jh}. \tag{2.7.10}$$

Element (k, k') of $\mathbf{H}_j'\mathbf{H}_h$ is the number of years with test j on plots k in the same year that test h is on plots k'.

The normal equations are as follows:

$$\boldsymbol{\Gamma}\begin{bmatrix}\hat{\mathbf{t}}\\ \hat{\mathbf{b}}\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{T}}\\ \hat{\mathbf{Z}}\end{bmatrix}, \tag{2.7.11}$$

where

$$r\boldsymbol{\Gamma} = \begin{bmatrix} rN\mathbf{C}_w - \mathbf{K}_t \mathbf{V}_p^{-1} \mathbf{K}_t' \sigma_p^2, & -\mathbf{K}_t \mathbf{V}_p^{-1} \mathbf{K}_b' \sigma_p^2 \\ -\mathbf{K}_b \mathbf{V}_p^{-1} \mathbf{K}_t' \sigma_p^2, & r(\mathbf{C}_w \times \mathbf{x}'\mathbf{x}) - \mathbf{K}_b \mathbf{V}_p^{-1} \mathbf{K}_b' \sigma_p^2 \end{bmatrix}, \tag{2.7.12}$$

$$r\hat{\mathbf{T}} = r\mathbf{T} - \mathbf{K}_t \mathbf{V}_p^{-1} \mathbf{P} \sigma_p^2, \tag{2.7.13}$$

and

$$r\hat{\mathbf{Z}} = r\mathbf{Z} - \mathbf{K}_b \mathbf{V}_p^{-1} \mathbf{P} \sigma_p^2. \tag{2.7.14}$$

The elements of $\boldsymbol{\Gamma}^{-1}$ give the variances and covariances of the estimated means and regressions on model I of Section 2.1, ignoring errors in the estimation of $\mathbf{V}^{-1}$. The modification given by equation (2.5.25) is required when model II is used.

When the cycles of the experiment are complete the unweighted means are efficient estimates of the elements $\boldsymbol{\tau}$ (Section 2.5.5). It follows that

$$\mathbf{K}_b \mathbf{V}_p^{-1} \mathbf{K}_b' = 0 \tag{2.7.15}$$

so that $\hat{\mathbf{b}}$ is given by

$$\mathbf{b} = r\{r(\mathbf{C}_w \times \mathbf{x}'\mathbf{x}) - \mathbf{K}_b \mathbf{V}_p^{-1} \mathbf{K}_b' \sigma_p^2\}^{-1} \hat{\mathbf{Z}}. \tag{2.7.16}$$

As $\sigma_p^2 \to \infty$

$$\hat{\mathbf{T}} \to \mathbf{T} - \mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{P} \tag{2.7.17}$$

and

$$\hat{\mathbf{Z}} \to \mathbf{Z} - \mathbf{K}_b \mathbf{K}_p^{-1} \mathbf{P}. \tag{2.7.18}$$

Hence within-plot estimates of the β_{jm} and phase differences, if any, between the τ_j can be obtained from

$$\mathbf{\Gamma}_w \begin{bmatrix} \mathbf{t}_w \\ \mathbf{b}_w \end{bmatrix} = \begin{bmatrix} \mathbf{T} - \mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{P} \\ \mathbf{Z} - \mathbf{K}_b \mathbf{K}_p^{-1} \mathbf{P} \end{bmatrix}, \tag{2.7.19}$$

with suitable restrictions on the elements of $\mathbf{t}_w$, where

$$\mathbf{\Gamma}_w = \begin{bmatrix} \mathbf{N}\mathbf{C}_w - \mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{K}_t', & -\mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{K}_b' \\ -\mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{K}_b', & (\mathbf{C}_w \times \mathbf{x}'\mathbf{x}) - \mathbf{K}_b \mathbf{K}_p^{-1} \mathbf{K}_b' \end{bmatrix}. \tag{2.7.20}$$

In a design with no phase differences

$$\mathbf{T} = \mathbf{K}_t \mathbf{K}_p^{-1} \mathbf{P}, \tag{2.7.21}$$

and only between-plot contrasts provide information on the τ_j. The within-plot analysis can be further restricted to within-year comparisons by substituting α'_{jh} for α_{jh} in the expression for $\mathbf{C}_w$ and multiplying the modified matrix by $\mathbf{I}_t - t^{-1}\mathbf{1}_t\mathbf{1}_t'$, as in Section 2.6.2.

2.8. *Further Work*

Further work is required on several problems of statistical technique in the type of analysis discussed above. Complications arise in the estimation of the errors of (1) experiments of phase-confounded design, (2) non-replicated experiments of basic or reduced design, and (3) any type of rotation experiment when the errors are heterogeneous or otherwise deviate from the two-component model of Section 2.2.

When a phase-confounded design is used the methods described above require extensive modification to allow for the increased complexity of the variance matrix of the $\bar{y}_{ij}$ when these have been adjusted for block differences. Some, but not all, of the plot and plot × year error degrees of freedom can be separated by the method of Section 2.4 when each block is replicated. The remaining error degrees of freedom usually can be separated only by fitting constants for plots, blocks, blocks × years, tests and tests × years simultaneously. Stevens (1956) discussed the analysis of a particular rotation experiment of phase-confounded design. He made many useful points but his analysis requires to be modified to distinguish between experimental errors and variations in the effects of the rotations from year to year and, for some contrasts, to allow for plot correlations.

The main problem in (2) arises in determining the separate estimates of σ_p^2 and σ_w^2 that are required to evaluate the errors of estimates of the $\bar{\tau}_j$ of equation (2.1.2) (model II). Methods, similar to that described by Patterson (1959) for a particular fixed-rotation experiment, may be useful in some circumstances. These methods are only available if $\sigma_{tys}^2 = 0$ for some contrasts. In theory unbiased estimates of the errors of the t_j (model I only) and b_{jm} of Section 2.5 can be obtained from between-plot and within-plot contrasts arising in the weighted analysis described in Sections 2.6 and 2.7.

Problem (3) may be important when a wide range of yields is produced by the different rotations. In one experiment in the Sudan Gezira, analysed by P. Roberts and D. K. Dutta Roy but not reported in the literature, there is clear evidence of heterogeneity of errors between different rotations. Heterogeneity of errors and other deviations from the two-component model can be allowed for fairly readily in experiments with rotations of equal length but suitable methods for rotations of differing lengths have not yet been developed. The possibility that an analysis of covariance on suitable plot variates of the type mentioned below may reduce heterogeneity of errors is worth noting.

Work is also required to develop methods for investigating fundamental relationships between crop yields and variates representing factors such as soil nutrients, weeds and disease, through which the rotations affect yields. These variates, which can be called plot variates, differ from the trend and seasonal variates considered in the present paper in that they vary between columns of the years × tests tables. Between-years contrasts must always be eliminated in estimating the relationships, or given appropriately small weight. This type of analysis is potentially much more useful than the analysis of trends, but its development and use have been limited by lack of suitable data.

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Discussion on Mr Patterson's Paper

Dr S. C. Pearce: It gives me great pleasure to propose a vote of thanks to Mr Patterson for the paper he has given us. For one thing, it is excellent both in presentation and in comprehensiveness: for another, he has dealt with a subject that badly needs study at the present time. A lot of experiments are necessarily of a long-term nature, e.g. those with certain kinds of animals and those on grassland so ancient "that the memory of man