In balanced 6×6 designs for nine treatments, only 2×2, 4×2, 2×4, 3×2 and 2×3 intercalates are possible; 3×3, 4×3 and 3×4 intercalates are not possible as no three rows or columns of a design have three treatments in common. In what follows, "4×2 and 2×4 intercalates" will be taken to exclude intercalates that are pairs of 2×2 intercalates.

A "family" of species is a group of species such that interchanging the rows or columns of any 2×2 intercalate in any species of the group generates a member of the group, and such that it is possible to pass from any member to any other by a succession of reversals of 2×2 intercalates. A "domain" of species is similarly defined, except that, in addition to interchanges of rows or columns in 2×2 intercalates, we now permit those interchanges of rows and columns in $n \times m$ intercalates which leave the same elements in each row and each column.

The species discovered by Kshirsagar and Pearce belong to the same domain, but not to the same family. No other domain is known.

Table 1 gives the distributions of the numbers of intercalates per species for the 344 species of the domain. Table 2 gives the distribution of the number of 2×2 intercalates per species for each of the 38 families. Table 3 gives the distribution of the numbers of $n \times m$ intercalates per species in the 16 species without any 2×2 intercalates.

No species in the domain has a 2×2 intercalate whose rows or columns may be interchanged without changing the species. There are, however, various species invariant under interchanges in other intercalates.

TABLE 1. DISTRIBUTIONS OF NUMBERS OF INTERCALATES PER SPECIES IN THE 344 SPECIES OF THE DOMAIN

	no. of intercalates in species									
	0	1	2	3	4	5	6	7	>7	total
2×2 intercalates	16	39	72	78	71	37	23	8	0	344
4×2 and 2×4 intercalates	159	112	42	22	8	0	1	0	0	344
3×2 and 2×3 intercalates	46	124	112	54	8	0	0	0	0	344

TABLE 2. DISTRIBUTION OF NUMBER OF 2×2 INTERCALATES PER SPECIES FOR EACH OF THE 38 FAMILIES IN THE DOMAIN

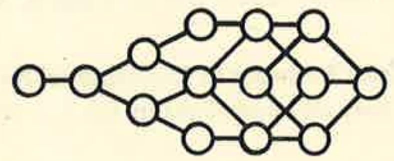
	no. of 2×2 intercalates in species								total no. of species
	0	1	2	3	4	5	6	7	
family (i)	1†	0	0	0	0	0	0	0	1†
each of families (ii)-(xvi)	1	0	0	0	0	0	0	0	1
each of families (xvii)-(xxii)	0	2	0	0	0	0	0	0	2
each of families (xxiii)-(xxvii)	0	2	1	0	0	0	0	0	3
family (xxviii)	0	0	0	4*	0	0	0	0	4*
family (xxix)	0	0	4	0	0	0	0	0	4
family (xxx)	0	1	3	1	0	0	0	0	5
family (xxxi)	0	1	5	1	0	0	0	0	7
family (xxxii)	0	1	3	5	0	0	0	0	9
family (xxxiii)	0	1	4	3	1	0	0	0	9
family (xxxiv)	0	2	3	4	1	0	0	0	10
family (xxxv)	0	1	2	8	2	1	0	0	14
family (xxxvi)	0	1	6	7	3	0	0	0	17
family (xxxvii)	0	2	16	8	22	5	5	1	59
family (xxxviii)	0	7	21	37	42	31	18	7	163

†This species has symmetry with respect to its three 3×2 (or 2×3) intercalates (see text).
*These species have symmetry with respect to their 2×2 intercalates (see text).

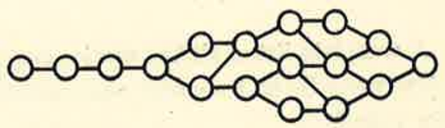
TABLE 3. DISTRIBUTIONS OF NUMBERS OF $n \times m$ INTERCALATES PER SPECIES IN THE 16 SPECIES WITHOUT 2×2 INTERCALATES

		no. of 3×2 and 2×3 intercalates in species				total
		0	1	2	3	
no. of 4×2 and 2×4 intercalates in species	0	0	3	2	3	8
	1	1	1	4	1	7
	2	0	0	1	0	1
total		1	4	7	4	16

All species except those of family (xxviii) (see Table 2) have the property that interchanges in different 2×2 intercalates of a species generate different species. Thus, for each family except family (xxviii), the relationships between the species can be illustrated by a diagram such as one of the following.



Family (xxxv)



Family (xxxvi)

In these diagrams, each circle represents a species. Emanating from each circle are as many paths as there are 2×2 intercalates in the corresponding species; two circles connected by a path correspond to two species either one of which can be obtained by interchanging the rows or columns of a 2×2 intercalate in the other.

Family (xxviii) contains four species each with three 2×2 intercalates and each with symmetry. Labelling the species A, B, C, D, an interchange in

(a) any 2×2 intercalate of A gives B;

(b) one 2×2 intercalate of B gives A, and either of the others gives C;

(c) one 2×2 intercalate of C gives D, and either of the others gives B;

(d) any 2×2 intercalate of D gives C.

A specimen member of species A is

$$\begin{array}{cccccc} 2 & 4 & 7 & 8 & 6 & 3 \\ 4 & 7 & 3 & 5 & 9 & 2 \\ 5 & 8 & 1 & 7 & 3 & 6 \\ 3 & 5 & 8 & 4 & 1 & 9 \\ 1 & 6 & 9 & 2 & 7 & 5 \\ 6 & 9 & 2 & 1 & 4 & 8 \end{array} \quad \ldots \quad (2)$$

The three independent 4×2 intercalates in (2) are in rows 1 and 2, 3 and 4, and 5 and 6. Interchanging rows in any of these 4×2 intercalates gives a species from family (xxxvii).

Of the sixteen species with no 2×2 intercalates, one, that of family (i) of Table 2, has three 3×2 (or 2×3) intercalates such that an interchange in any one of them gives the same species from family (xxxviii). A specimen member of family (i) is

3	4	8	7	6	2
4	7	9	2	3	5
6	5	3	1	7	8
5	9	1	8	4	3
2	6	7	5	1	9
1	8	2	4	9	6

... (3)

The 3×2 intercalates in (3) are not independent; they are in rows 4 and 2, 4 and 3, and 4 and 6.

A complete list of all 344 species may be had from Rothamsted Experimental Station. The Rothamsted Orion computer was used to check that all intercalates in the 344 species had been found.

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BALANCED 6×6 DESIGNS FOR 9 TREATMENTS

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SUMMARY. Kshirsagar (1957) and Pearce (1963) gave two different balanced 6×6 designs for nine treatments. These designs belong to a domain of 344 species, grouped into 38 families. The methods used to classify and enumerate the species are immediate extensions of those used previously for Latin squares and rectangles.

Using the digits 1, 2, ..., 9 to denote treatments, the row-and-column design

2	4	8	7	6	3
3	5	7	2	4	9
5	6	3	1	7	8
4	9	1	8	3	5
6	7	9	5	1	2
1	8	2	4	9	6

... (1)

due to B. V. Shah, and published by Kshirsagar (1957), is balanced in the sense that the variance of an estimated treatment difference is independent of the treatments compared. A similar design was given by Pearce (1963).

Another arrangement, with the same elements in its rows and columns as (1), but possibly with rows and columns in different orders, can be obtained by applying to the treatments in (1) the permutation

$$P_1 = (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$$

or

$$P_2 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$$

or

$$P_3^2 \text{ where } P_3 = (2\ 8\ 4\ 5\ 3\ 6\ 7\ 9)$$

or any product of these. An arrangement with the same elements in its rows (columns) as in the columns (rows) of (1) can be obtained by applying permutation P_3 to the treatments in (1). All these arrangements will be said to belong to the same "species", because they can be obtained from one another by permutations of rows, columns and treatments.

Following the definitions of Norton (1939) for Latin squares, and of Preece (1966) for Youden rectangles and other Latin rectangles, we define a "$n \times m$ intercalate" in a design such as (1) as an $n \times m$ Latin rectangle embedded in the design. Thus rows 1 and 3 of (1) contain the 2×2 intercalate

8	3
3	8

and rows 2 and 5 contain the 4×2 intercalate

5	7	2	9
7	9	5	2