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General Balance and Treatment Permutations

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ABSTRACT

Many equi-replicate designs, whether ordinary block designs or multi-stratum designs, have a structure which is left unchanged by a group G of permutations of the treatments. This group G is often used in the construction of the design. A survey of the properties of such designs is given, and useful necessary conditions on G are found for such a design to be generally balanced. If G is Abelian, then the necessary conditions are satisfied: in this case explicit formulas for the efficiency factors are given.

1. INTRODUCTION

The importance and utility of the property of general balance in a designed experiment have been explicated by Houtman and Speed [41] and Nelder [62]. It is thus not surprising that most designs in common use, and many that arise from the large number of constructions available in the literature, are generally balanced. In this paper we shall show that certain symmetry conditions on the design imply general balance and that, therefore, certain methods of construction will always produce generally balanced

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designs. As Houtman and Speed showed, if a design has only a single partition of the plots into blocks, then the design will be generally balanced no matter what allocation of treatments is used. Although our methods shed further light on such designs, we are principally concerned with more complex designs, in particular those with a Tjur block structure.

Section 2 introduces general balance, and Section 3 describes, with examples, the important role which groups of symmetries play in many designs. This role is developed further in Section 4, whilst Section 5 uses the theory of group characters to give an important and useful criterion on the group of symmetries which guarantees that the design is generally balanced. The particular case of generalized cyclic designs (in the sense of [49]), in which the group is commutative (Abelian), is treated (independently of Section 5) in the final two sections of the paper. For this case we give explicit, practical formulas for the efficiency factors and projection matrices needed to analyse designs of this type. Concepts from permutation-group theory, such as transitivity, which are used without comment can be found in the first chapter of [81], whilst those from linear algebra are in [37].

2. GENERAL BALANCE

We start with a definition (from [41]) of general balance which is so general that it does not even require a block structure. Let Ω be a set of plots. Let $\bigoplus_{\alpha\in \mathrm{A}}\mathscr{S}_\alpha$ be an orthogonal direct-sum decomposition of the real vector space $\mathbf{R}^\Omega$, and let S_α denote the matrix of the orthogonal projection onto $\mathscr{S}_\alpha$. Let $(y_\omega)_{\omega\in\Omega}$ be random variables and suppose that $\mathrm{Cov}(y)=\sum_{\alpha\in \mathrm{A}}\xi_\alpha S_\alpha$, where the S_α are as above and are *known*, but the ξ_α may be unknown (see [8, 61]). The subspaces $\mathscr{S}_\alpha$ are called *strata*, and the matrices S_α are called *stratum projection matrices*.

Let T be a set of treatments. Suppose that treatments are allocated to plots according to a map $\phi:\Omega\to T$ whose dual is represented by the $\Omega\times T$ design matrix X: thus the (ω,t) entry of X is defined by

$$X(\omega,t)=\begin{cases}1 & \text{if } \phi(\omega)=t,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose further that there is a positive integer r (called the *replication*) such that $|\phi^{-1}(t)|=r$ for all t in T.

In this paper a *design* is a quadruple $(\Omega,\{\mathscr{S}_\alpha:\alpha\in \mathrm{A}\},T,\phi)$ with the properties listed above. In particular, all the designs are equireplicate.

For each α, denote by L_α the matrix $X'S_\alpha X$, which is the *information matrix* for the stratum $\mathscr{S}_\alpha$. Suppose that $\oplus_{j\in J} W_j$ is a direct-sum decomposition of $\mathbf{R}^T$ (not necessarily orthogonal). In [62, 63] Nelder defined a design to be *generally balanced with respect to* $\oplus_{j\in J} W_j$ if, for all α in A and all j in J, the space W_j is a subspace of an eigenspace of L_α. He called the eigenvalue $\lambda_{\alpha j}$ of $r^{-1}L_\alpha$ on W_j the *efficiency factor* for W_j in stratum $\mathscr{S}_\alpha$, and showed that $0 \leqslant \lambda_{\alpha j} \leqslant 1$ and $\Sigma_{\alpha\in \mathrm{A}}\lambda_{\alpha j} = 1$ for all j in J.

A design is *generally balanced* if there exists a decomposition of $\mathbf{R}^T$ with respect to which it is generally balanced; in this case there is a unique coarsest decomposition of $\mathbf{R}^T$ with respect to which it is generally balanced. The subspaces in this decomposition are all the nonzero intersections of the form $\bigcap_{\alpha\in \mathrm{A}} E_\alpha$, where E_α is an eigenspace of L_α. This decomposition is coarsest in the sense that any other decomposition with respect to which the design is generally balanced may be obtained by further decomposing one or more of the spaces therein.

More simply, a design is generally balanced if and only if there is a basis of $\mathbf{R}^T$ whose elements are eigenvectors of every information matrix. We therefore make the following definition.

DEFINITION. A set of real matrices is *cospectral* if there is a basis of $\mathbf{R}^T$ whose elements are eigenvectors of every matrix in the set.

Thus a design is generally balanced if and only if its set of information matrices is cospectral. However, it is difficult to test, in general, whether a set of matrices is cospectral; so in Theorem 2.2 we give a useful and simple test for general balance. We first need the following definition and lemma, which proves the theorem.

DEFINITION. A set of matrices is *commutative* if every pair M, N of matrices in the set commutes (that is, $MN = NM$).

LEMMA 2.1. *A set of diagonalizable matrices is commutative if and only if it is cospectral.*

Proof. See the proof of Theorem IV.7 of [44] or the results in Sections I.49–50 of [82]. ∎

THEOREM 2.2. *A design is generally balanced if and only if the set of information matrices* $(L_\alpha)_{\alpha\in \mathrm{A}}$ *is commutative.*

In an ordinary block design there is a single nontrivial partition, β, of Ω into equal-sized blocks; the two trivial partitions are ε, whose blocks are the singleton plots, and μ, whose only block is the whole of Ω. There are three strata: $\mathscr{S}_\mu$ consists of constant vectors; $\mathscr{S}_\beta$ and $\mathscr{S}_\varepsilon$ are the interblock and intrablock strata, respectively. Thus $A = \{\mu, \beta, \varepsilon\}$, and the three information matrices can be calculated as follows:

$$L_\mu = rn^{-1}J,$$

$$L_\beta = k^{-1}C - rn^{-1}J,$$

$$L_\varepsilon = rI - k^{-1}C,$$

where $n = |T|$, k is the size of each block, I is the $n \times n$ identity matrix, J is the $n \times n$ all-1s matrix, and C is the concurrence matrix. Because C has constant row and column sums, these three information matrices commute. Thus all ordinary block designs are generally balanced; this proof is essentially the same as that given in Section 5.4 of [41], where the authors go on to point out that this result gives no help whatever in finding the relevant decomposition of $\mathbf{R}^T$. However, for the class of designs we consider in this paper, we shall give this decomposition in a very explicit form.

Moreover, our results apply to a richer class of structures, those defined by Tjur [80, Section 4]. A *Tjur block structure* is a semilattice Γ of partitions of Ω (called *block systems* on Ω), with various conditions on Γ. Each partition γ is into equal-sized blocks of size k_γ; and Γ contains the trivial partition ε (singletons). (Most Tjur designs in practical use also include the other trivial partition μ.) The remaining conditions imposed by Tjur on Γ are designed to ensure that Γ indexes a particular orthogonal direct-sum decomposition $\bigoplus_{\gamma \in \Gamma} \mathscr{S}_\gamma$ of $\mathbf{R}^\Omega$: the construction of its projection matrices is given in the next paragraph.

Each partition γ defines a *blocks-averaging matrix* B_γ on $\Omega \times \Omega$: the (ω, θ) entry of B_γ is k_γ^{-1} if ω and θ are in the same γ-block; otherwise it is zero. Denote by C_γ the matrix $k_\gamma X' B_\gamma X$, which is the concurrence matrix for the block system γ. We define a map $z: \Gamma \times \Gamma \to \{0, 1\}$ by

$$z(\gamma, \alpha) = \begin{cases} 1 & \text{if } \alpha \text{ nests } \gamma \text{ (that is, each } \gamma\text{-block is contained in an } \alpha\text{-block)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $Z = [z(\gamma, \alpha)]$ has an inverse M (see [3, Chapter IV] and

[77]), and

$$B_\gamma = \sum_{\alpha \in \Gamma} z(\gamma, \alpha) S_\alpha, \tag{1}$$

$$S_\alpha = \sum_{\gamma \in \Gamma} m(\alpha, \gamma) B_\gamma, \tag{2}$$

where $M = [m(\gamma, \alpha)]$. (The function m is the Möbius function of the semilattice Γ.)

We call a design $(\Omega, \{\mathscr{S}_\alpha : \alpha \in \mathrm{A}\}, T, \phi)$ a *Tjur design* if A can be identified with the semilattice Γ of a Tjur block structure in such a way that the stratum projection matrices $(S_\alpha)_{\alpha \in \mathrm{A}}$ are precisely those given by Equation (2). We shall abuse our notation slightly and write such a Tjur design as $(\Omega, \Gamma, T, \phi)$. The class of Tjur designs includes most of the equireplicate designs in practical use or in the literature. This is because the class of Tjur block structures includes many, possibly all, of the classes of block structures studied by the other authors: for example, the *simple orthogonal block structures* [61], the *poset block structures* or *distributive block structures* [77, 9, 11, 16], the *orthogonal block structures* [77, 14], the *complete balanced response structures* [79, 55], and the *group block structures* [12, 14].

The above linear relationships, (1) and (2), between the matrices $(B_\gamma)_{\gamma \in \Gamma}$ and $(S_\gamma)_{\gamma \in \Gamma}$ give us the following characterization of general balance for Tjur designs.

THEOREM 2.3. *A Tjur design is generally balanced if and only if the set of concurrence matrices $(C_\gamma)_{\gamma \in \Gamma}$ is commutative.*

Proof. Since $k_\gamma^{-1} C_\gamma = X' B_\gamma X$ and $L_\gamma = X' S_\gamma X$, Equations (1) and (2) give $(C_\gamma)_{\gamma \in \Gamma}$ as linear combinations of $(L_\gamma)_{\gamma \in \Gamma}$ and vice versa. Thus the concurrence matrices $(C_\gamma)_{\gamma \in \Gamma}$ commute with each other if and only if the information matrices $(L_\gamma)_{\gamma \in \Gamma}$ do so. Hence Theorem 2.3 is a corollary of Theorem 2.2 ∎

3. TREATMENT PERMUTATIONS

In Section 5 we shall use our characterization of general balance in terms of the concurrence matrices of a design (Theorem 2.3) to prove that certain symmetry conditions on these matrices are sufficient to ensure general balance. These symmetry conditions are best expressed in terms of a group of permutations of the set G of treatments; they make precise the idea that these permutations of the treatments do not alter the design (that is, the relationship between ϕ and the block structure) in any way which would affect the analysis of the design. We shall further show that this general balance is with respect to a decomposition of $\mathbf{R}^T$ which is determined by the group of permutations, and give explicit formulas for the projectors onto the subspaces in this decomposition.

In practice, this group of permutations is often a group of symmetries of the inherent structure of the set T of treatments. In this case it is likely that the design is generally balanced with respect to a decomposition of $\mathbf{R}^T$ related to this structure; this can greatly aid interpretation of the analysis (see [68, Chapter 4]). The structure of T may rise naturally from the problem under investigation (see Example 5.3 below) or may be an artifice introduced to aid the construction of the design as, for example, the use of affine geometries in the construction of classical factorial designs (see [20, 21]). Such aids to construction are the motivation for much of the material in subsequent sections of this paper.

Let G be a group of permutations of the set T of treatments. Following [81], we write the image of a treatment t under a permutation g as t^g. It is then necessary to write the elements of $\mathbf{R}^T$, which are functions from T to $\mathbf{R}$, on the *right* of their arguments (thus, for t in T and v in $\mathbf{R}^T$, we write tv for the "t entry in the vector v"). Each permutation g of T defines a linear transformation P_g of $\mathbf{R}^T$ by

$$t(vP_g) = (t^{g^{-1}})v \qquad \text{for } t \text{ in } T \text{ and } v \text{ in } \mathbf{R}^T, \tag{3}$$

and the map $g \mapsto P_g$ is a faithful linear representation of G on $\mathbf{R}^T$ (see [73, Section 1.2]), which is called the *permutation representation* of G on $\mathbf{R}^T$. The matrix of P_g with respect to the natural basis of $\mathbf{R}^T$ is the *permutation matrix* for g: the (t, u) entry of this matrix is equal to 1 if $t^g = u$ and to 0 otherwise. There should be no confusion if this matrix is also denoted P_g.

DEFINITION. Let G be a group of permutations of a set T of size n. An $n \times n$ matrix *centralizes* G if it commutes with the matrices P_g for all g in G. The set of all (real) matrices which centralize G is called the (real) *centralizer algebra* of G. (See [81, Section 28]: although Wielandt calls this algebra the centralizer *ring*.)

Note that a matrix M commutes with a permutation matrix P_g if and only if applying the permutation g to both the rows and columns of M does not change M.

For an ordinary block design, with $\Gamma = \{\mu, \beta, \varepsilon\}$ as in Section 2, Sinha [76] defined a design $(\Omega, \Gamma, T, \phi)$ to be *simple* if there is a transitive group G of permutations of T such that L_β centralizes G. Since I and J both centralize every permutation group G, the design is simple if and only if there is a transitive group G which is centralized by all three concurrence matrices, C_μ, C_β, and C_ε. We want to extend Sinha's definition to Tjur designs, and make it specific to the group G.

DEFINITION. A Tjur design $(\Omega, \Gamma, T, \phi)$ is *G-central* if G is a group of permutations of T such that the concurrence matrices $(C_\gamma)_{\gamma \in \Gamma}$ all centralize G.

Thus a design is G-central if and only if each of its concurrence matrices is unchanged by each of the permutations in G, applied to both rows and columns.

Any condition on the permutation group G which ensures that the set of concurrence matrices of a G-central design is cospectral will ensure that all such designs are generally balanced. The concurrence matrices of a G-central design are real symmetric matrices in the centralizer algebra of G. We therefore make the following definition.

DEFINITION. A permutation group G is *cospectral* if the set of real symmetric matrices in the centralizer algebra of G is cospectral.

Conditions for a group to be cospectral have been studied in many contexts, including other aspects of the design of experiments: see, for example, [60, 38, 45, 28, 26, 29]. One of the major purposes of this paper is to describe a property of the permutation group G which is equivalent to its being cospectral and which is particularly useful in that it leads to formulas for the projectors onto, and dimensions of, the relevant subspaces: these formulas can be used to specify an appropriate analysis of variance and calculate its efficiency factors, given sufficient information about the group G. This property is in fact a condition on the permutation representation of the group on $\mathbf{R}^T$, and its description involves the theory of group characters: we shall therefore leave this until Section 5.

We conclude this section with some examples of G-central designs chosen to illustrate certain more straightforward properties of the permutation group G (including that of being cospectral) and their relationship to the general balance of the design.

EXAMPLE 3.1. Identify T with the symmetric group S_3, and let G also be equal to S_3, with the *right regular action* on T: that is, $t^g = tg$. We may

TABLE 1

DESIGN FOR EXAMPLE 3.1

[1	a]	[1	b]
[a	a^2]	[a	a^2b]
[a^2	1]	[a^2	ab]
[b	ab]	[b	1]
[a^2b	b]	[a^2b	a]
[ab	a^2b]	[ab	a^2]

write the elements of T as 1, a, a^2, b, ab, a^2b, where $a^3 = b^2 = 1$ and $ba = a^2b$. The design has two block systems, α and β, with six α-blocks of size 4, each of which contains two β-blocks of size 2: thus $|\Omega| = 24$. Treatments are allocated to plots as in Table 1 (where square brackets enclose the β-blocks and rows are α-blocks). With the elements of T in the order given above, the concurrence matrices are as follows:

$$C_\alpha = \begin{bmatrix} 6 & 2 & 2 & 4 & 2 & 0 \\ 2 & 6 & 2 & 2 & 0 & 4 \\ 2 & 2 & 6 & 0 & 4 & 2 \\ 4 & 2 & 0 & 6 & 2 & 2 \\ 2 & 0 & 4 & 2 & 6 & 2 \\ 0 & 4 & 2 & 2 & 2 & 6 \end{bmatrix}, \qquad C_\beta = \begin{bmatrix} 4 & 1 & 1 & 2 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 2 \\ 1 & 1 & 4 & 0 & 2 & 0 \\ 2 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 2 & 1 & 4 & 1 \\ 0 & 2 & 0 & 1 & 1 & 4 \end{bmatrix}.$$

These both centralize G (this may be checked directly; it also follows from Theorem 4.2). However, these two concurrence matrices do not commute (check the $(a, 1)$ entries), so, by Theorem 2.3, the design is not generally balanced (nor is G cospectral). Since G is both transitive and regular in this case, this example shows that neither transitivity nor even regularity of G suffices to ensure general balance of G-central designs.

By contrast, if G is transitive *and Abelian* (as is the case in many important applications of these methods) then, as we show in Section 7, G is cospectral and so general balance is assured. (Note that, for an Abelian permutation group, transitivity is equivalent to regularity.)

EXAMPLE 3.2. If, in Example 3.1, the α-blocks (or the β-blocks) alone are considered, then the design is an ordinary block design and is thus necessarily generally balanced even though G is not cospectral. This emphasizes the fact that the cospectral condition is *sufficient* for general balance but is *not* equivalent to it.

EXAMPLE 3.3 (See [76, Section 2.3(b)(i)]). Suppose that a design is G-central and that the group G is 2-transitive on T. Then every matrix in the centralizer algebra is a linear combination of the matrices I and J, so G is cospectral and the design is generally balanced. This is a rather trivial sort of general balance, for the design is now totally balanced (in the usual sense) for each block system separately: in other words, each nonzero information matrix has eigenspaces W_0 and $W_0^{\perp}$, where W_0 consists of the constant vectors in $\mathbf{R}^T$. See [70, 72, 75, 78, 1] for some examples, and general constructions, of designs which are balanced incomplete block designs (BIBDs) with respect to each block system separately.

4. GROUP-GENERATED BLOCKS

The most common way in which G-central designs arise, in practice, is a direct consequence of a frequently used method of constructing designs: an allocation of treatments to one, or more, *initial blocks* is first made, and then the allocations to further blocks are calculated by applying each permutation in G to each initial block. This method was used to construct the design in Example 3.1, starting from the two initial β-blocks $\{1, a\}$ and $\{1, b\}$; these were combined to give the single initial α-block $\{1, a, 1, b\}$.

This method of constructing designs has a history which goes back at least as far as the cyclic and dicyclic designs in [35, Table XVII]; see also the tables in [53]. These tables are for designs with one nontrivial block system, as is much of the recent use of this method to construct generalized cyclic designs (see, for example, [49, 32]). However, some authors have used these methods to produce designs with somewhat richer block structures [70, 71, 30, 40, 75, 4, 78, 1, 2, 52, 36, 43]. Another important application of these methods is to factorial designs; their use here ranges from the classical work [21, 34] for a single nontrivial block system and a prime-power number of treatments, to the more recent results [15, 7, 14] in which the block structure can be any group block structure and the number of treatments is arbitrary. The addition of a pseudo-factorial structure to an unstructured set of treatments enables the methods of construction and analysis for factorial designs to be used in a far wider context.

In fact, Bose [19] seems to have been the first to replace a regular cyclic group by a regular general Abelian group in this method of construction. He called his method "the method of differences." Bose and Nair [23] showed that this method need not be restricted to balanced designs. Bruck [25] extended the method to any regular permutation group.

The history of the term "generalized cyclic" needs explanation, as it has been given two different meanings. In [54, Chapter 13], P. W. M. John described Bose's method of differences, calling the ensuing designs "cyclic" whether or not the Abelian group is cyclic. In [54, Chapter 15] he gave the identical construction, but with slightly different notation, and called the designs "generalized cyclic" on the grounds that Abelian groups could be considered to be a generalization of cyclic groups. J. A. John [49] used P. W. M. John's new terminology and made it widely known.

In all of the examples that we have mentioned so far in this section, the set T of treatments is identified with the group G, which acts regularly on itself: for cyclic designs, G is a cyclic group; for dicyclic designs, G is the direct product of two cyclic groups; for classical factorial designs, G is the additive group of a finite field. These are all Abelian groups, so these designs are all, in some sense, special cases of the generalized cyclic designs and general factorial designs, in which G can be any Abelian group.

However, there is another obvious way to generalize the class of regular cyclic permutation groups: this is to remove the regularity condition. Patterson and Williams [67] and Jarrett and Hall [47, 48] give a construction which is somewhat different from the examples mentioned so far but which nevertheless follows the pattern outlined at the beginning of this section. They identify T with a cyclic group, but they permit G to be a proper subgroup of T, so that G is merely a semi-regular group of permutations of T; this is equivalent to identifying T with a union of distinct copies of G. Jarrett and Hall call their designs "generalized cyclic," but their construction can be used to produce *any* design by taking G to be the trivial group; therefore no new nomenclature is needed for this class of designs, and the term "generalized cyclic" could with advantage be restricted to the class of designs given by John [49]. This is not to detract from the utility of Jarrett and Hall's method of construction, particularly when $|T|/|G|$ is small.

The group-generation method of constructing and describing a design has also been much used by mathematicians in their never ending, and recently most productive, search for BIBDs (see, for example, [6, 59, 46]).

Before discussing properties of designs constructed in this way we need to describe these designs in a way which is based closely on this construction method but is sufficiently rigorous to enable precise statements about them to be made and proved.

One example of the need for a rigorous approach is the fact that blocks are subsets of Ω, so a block may well contain a treatment, or treatments, more than once, and two different blocks may have identical allocation of treatments. Thus we cannot identify blocks with subsets of T and must take some care with our definitions. We represent multiple occurrences of a treatment within a block by using *multisets* [33]: a multiset is just a function from T to the natural numbers $\mathbf{N}$ which records the number of occurrences of each treatment in a given block. Thus if b is a block, then its corresponding multiset is the function $K_b: T \to \mathbf{N}$ defined by

$$K_b(t) = |\{\omega \in b: \phi(\omega) = t\}| \qquad \text{for } t \text{ in } T.$$

The action of the group G on T extends naturally to an action of G on multisets by

$$K^g(t) = K(t^{g^{-1}}) \qquad \text{for g in } G.$$

In this action the *stabilizer* of a multiset K is the subgroup $\{g \in G: K^g = K\}$, and the *orbit* containing K is the set of distinct multisets K^g, $g \in G$. We now need several definitions.

DEFINITION. Let Δ be the design $(\Omega, \Gamma, T, \phi)$. Let γ in Γ be a block system on Ω, and let G be a group of permutations of T. The design Δ is a *thin* (G, γ)*-design* if (i) no two γ-blocks have the same multiset and (ii) the set of multisets of the γ-blocks forms a single orbit of G.

Let Ω' be a subset of Ω which is a union of γ-blocks. Denote by γ' the partition of Ω' into these γ-blocks and by ε' the partition of Ω' into singletons; and let Δ' be the design $(\Omega', \{\gamma', \varepsilon'\}, T, \phi)$. The subset Ω' is a (G, γ)*-component* of Ω if Δ' is a thin (G, γ')-design.

The subset Ω'' of Ω is a *homogeneous* (G, γ)*-part* of Ω if Ω'' is a maximal disjoint union of (G, γ)-components which have the same set of multisets.

The design Δ is a (G, γ)*-design* if Ω is a disjoint union of (G, γ)-components. Finally, Δ is a *homogeneous* (G, γ)*-design* if Ω is a homogeneous (G, γ)-part of itself.

Less formally, a thin design is one whose blocks are generated from a single initial block by applying each element of G in turn to the initial block and using all the *distinct* blocks which arise (so that no two distinct blocks have the same treatment allocation). Let H be the stabilizer of the multiset of the initial block. When H is nontrivial some authors (such as John [49], Jarrett and Hall [47], Dean and Lewis [32]) say that the thin design generated by this initial block has a *fractional set* of blocks, whilst the design which has a block for each g in G whether or not the multisets are distinct is said to have a *full set* of blocks. Since G is a group, in the latter, "full," design every multiset of the thin design occurs as the multiset of exactly $|H|$ blocks, so the "full" design is the disjoint union of $|H|$ identical thin designs and is thus a homogeneous design.

Note that the division of a (G, γ)-design into homogeneous (G, γ)-parts is unique, whereas the division into (G, γ)-components may not be.

DEFINITION. A *G-design* is a design $(\Omega, \Gamma, T, \phi)$ which is a (G, γ)-design for all γ in Γ.

Having clarified the terminology, we shall now show that a G-design is G-central; but this requires some further definitions.

DEFINITION. Let $\gamma \in \Gamma$. A permutation h of Ω is a *γ-morphism* of the design $(\Omega, \Gamma, T, \phi)$ if, for all ω and θ in Ω,

(i) ω, θ are in the same γ-block $\Leftrightarrow$ ω^h, θ^h are in the same γ-block;
(ii) $\phi(\omega) = \phi(\theta)$ $\Leftrightarrow$ $\phi(\omega^h) = \phi(\theta^h)$.

A permutation h of Ω is an *automorphism* of the design $(\Omega, \Gamma, T, \phi)$ if it is a γ-morphism for all γ in Γ.

Thus any group of γ-morphisms acts on T and on the set of γ-blocks.

THEOREM 4.1. *Let Δ be the design $(\Omega, \Gamma, T, \phi)$, let $\gamma \in \Gamma$, and let G be a group of permutations of T. Then the following conditions are equivalent*:

(i) *There is a group of γ-morphisms of Δ whose action on T is isomorphic to G.*
(ii) *The design Δ is a (G, γ)-design.*
(iii) *There is a group of γ-morphisms of Δ whose action on T is faithful and isomorphic to G.*

Proof. (i) $\Rightarrow$ (ii): Let $\hat{G}$ be such a group of γ-morphisms, and let $\psi: \hat{G} \to G$ be the epimorphism induced by the action of $\hat{G}$ on T. If b is a γ-block and $h \in \hat{G}$, then b^h is also a γ-block, and its multiset is $K_b^{\psi(h)}$. Let G_b and $\hat{G}_b$ be the stabilizers of K_b in G and of b in $\hat{G}$ respectively. Then $\hat{G}_b \subseteq \psi^{-1}(G_b) \subseteq \hat{G}$. Let Ω' be the union of the γ-blocks b^h for h in $\hat{G}$. Then the number of γ-blocks in Ω' is $|\hat{G}|/|\hat{G}_b|$, and their multisets are the multisets in the orbit of K_b, each occurring $|\psi^{-1}(G_b)|/|\hat{G}_b|$ times. Thus $(\Omega', \{\gamma', \varepsilon'\}, T, \phi)$ is a homogeneous (G, γ)-design. Since Ω is the disjoint union of such subsets Ω', the design Δ is a (G, γ)-design.

(ii) $\Rightarrow$ (iii): Suppose that Δ is a (G, γ)-design. Since this implication holds for the whole of Δ provided that it holds (with respect to γ') for each component, it suffices to consider the case when Δ is thin. We need to show that there is a faithful action of G on Ω as a group of γ-morphisms which is consistent with the action of G on T. Label each plot ω in Ω by the triple $(K_b, \phi(\omega), l)$, where b is the block containing ω and l is an integer, $1 \leqslant l \leqslant K_b(\phi(\omega))$, this integer l being used to distinguish between the plots in a block which have the same treatment. Since Δ is thin, the multiset K_b

identifies the block b uniquely, so each such triple labels a unique plot. Let $g \in G$; then (K^g, t^g, l) is such a triple if and only if (K, t, l) is, so we can define an action of g on Ω by

$$(K,t,l)^g = (K^g, t^g, l).$$

Then g is a γ-morphism. Moreover, this defines an action of the group G on Ω, because

$$\left((K,t,l)^g\right)^h = (K,t,l)^{gh}$$

for g and h in G. This action is faithful, since $\phi(\Omega) = T$ and a plot cannot be allocated two distinct treatments.

(iii) $\Rightarrow$ (i): Obvious. ∎

THEOREM 4.2. *Suppose that, for some γ in Γ, the design $(\Omega, \Gamma, T, \phi)$ is a (G, γ)-design. Then the concurrence matrix C_γ centralizes G.*

Proof. For t, u in T define the subset $\Omega(t,u)$ of $\Omega \times \Omega$ by

$$\Omega(t,u) = \begin{array}{l}\{(\omega,\theta) \in \Omega \times \Omega : \phi(\omega) = t, \phi(\theta) = u, \\ \quad \text{and } \omega \text{ and } \theta \text{ are in the same } \gamma\text{-block}\}\end{array}.$$

Then the concurrence of t and u in γ-blocks is just $|\Omega(t,u)|$. By Theorem 4.1, G acts on Ω as a group of γ-morphisms. This action thus induces an action of G on $\Omega \times \Omega$ for which

$$(\Omega(t,u))^g = \Omega(t^g, u^g)$$

for all t and u in T and all g in G. Hence the (t,u) and (t^g, u^g) entries of C_γ are the same for all g in G. Thus C_γ centralizes G. ∎

The converse of Theorem 4.2 is false, as the following example shows.

EXAMPLE 4.1. Let $|\Omega| = 21$ and $\Gamma = \{\mu, \gamma, \varepsilon\}$, where γ has seven blocks of size 3. Let $T = \{1,2,\ldots,7\}$ and let G_1 and G_2 be the cyclic groups generated by (1234567) and (1324567) respectively. Let Δ be the thin G_1-design with initial block $\{1,2,4\}$ so that Δ is isomorphic to the projective plan over GF(2). Then $C_\gamma = 2I + J$, so C_γ centralizes the whole symmetric group S_7 on T: in particular, C_γ centralizes G_2. But Δ is not a G_2-design.

Sinha [76] gives a more extreme counterexample. Let Δ be any BIBD whose automorphism group is trivial, and let G be a nontrivial group of permutations of T. Then Δ is G-central but Δ is not a G-design.

On the other hand, Theorem 4.2 shows that every G-design is G-central, so the following result is an immediate consequence of the definition of cospectral.

THEOREM 4.3. *If G is cospectral, then every G-design is generally balanced.*

To a combinatorialist or algebraist, it might be more natural to define a G-design to be a design which has a group of automorphisms inducing G on T. Such a design certainly *is* a G-design in our sense, and so is generally balanced provided G is cospectral. Although most families of G-designs which we have seen in the literature do have such groups of automorphisms, a G-design does not necessarily possess a group of automorphisms inducing G on T, as the next two examples (one involving crossing and the other nesting) show. Thus, being a G-design is a strictly weaker property, but it is all that is required of a design to guarantee general balance (for cospectral G).

EXAMPLE 4.2. Let $|\Omega| = n^2$ and $\Gamma = \{\mu, \rho, \sigma, \varepsilon\}$, where the ρ-blocks and σ-blocks respectively are the rows and columns of an $n \times n$ array. Let $T = \{1, 2, \ldots, n\}$ with ϕ giving a Latin-square design, and let G be the symmetric group S_n on T. Then this design is a (G, ρ)-design and a (G, σ)-design. However, for almost all choices of ϕ there is no group of automorphisms of the Latin-square design which induces G on T.

For example, if $n = 5$, then the Latin square given by ϕ lies in one of the two transformation sets given in Table XV of [35]. Thus it is isotopic either to the square in Table 2 or to the well-known cyclic Latin square: in the former case the automorphism group of the design induces on T the alternating group A_4 (of order 12), fixing treatment 1; in the latter case the group induced on T is the affine group AGL(1,5) (of order 20).

EXAMPLE 4.3. Let $|\Omega| = 20$ and $\Gamma = \{\mu, \alpha, \beta, \varepsilon\}$, where α has five α-blocks of size 4, each containing two β-blocks of size 2. Let $T = \{0, 1, \ldots, 4\}$, and let G be the cyclic group on T generated by $(0\,1\,2\,3\,4)$. The design is in Table 3 (here square brackets enclose β-blocks and rows are α-blocks); it is a (G, α)-design and a (G, β)-design. Its automorphism group acts faithfully

TABLE 2

LATIN SQUARE FOR EXAMPLE 4.2

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1

on T. Although this action is isomorphic to that of the affine group AGL(1,5) on T, no element of it acts on T as $(0\,1\,2\,3\,4)$. Thus there is no group of automorphisms of the design whose action on T is that of G.

It is well known that the concurrence matrix of a thin cyclic block design is circulant and that its first row can be calculated from an examination of the initial block, without constructing the whole design. Likewise, if G is any permutation group, the calculation of the concurrences of a G-design can be simplified: our next theorem gives the details. We need two more pieces of notation, one for permutation groups and one for multisets. First, for any t, u in Ω, the subgroup G_{tu} is the pointwise stabilizer in G of t and u. Secondly, the usual notion of a tensor product of functions, whose domain is the direct product of the domains of the individual functions, gives, for multisets,

$$\begin{aligned}(K_1 \otimes K_2)(t,u) &= K_1(t) \times K_2(u) \\ &= |\{(\omega,\theta) \in b_1 \times b_2 : \phi(\omega) = t \text{ and } \phi(\theta) = u\}|\end{aligned}$$

if K_1 and K_2 are the multisets of the blocks b_1 and b_2 respectively.

THEOREM 4.4. *Let Δ be a thin (G,γ)-design with initial block b. Let K be the multiset of b, and let H be the stabilizer in G of K. For each orbit U of G on $T \times T$ put $n_U = \Sigma(K \otimes K)(t,u)$, the sum being over pairs (t,u) in U. Then, for any pair (x,y) in U, the concurrence of x and y in γ-blocks is equal to $n_U|G_{xy}|/|H|$.*

TABLE 3

DESIGN FOR EXAMPLE 4.3

[0	1]	[2	3]
[2	0]	[4	1]
[1	3]	[0	4]
[4	2]	[3	0]
[3	4]	[1	2]

Proof. The number of γ-blocks in the design is $|G|/|H|$, and each γ-block contributes n_U concurrences of pairs (t, u) in U. This gives a total of $n_U|G|/|H|$ such concurrences. By the proof of theorem 4.2, all pairs in U have the same concurrence. The number of such pairs is equal to $|G|/|G_{xy}|$, so the concurrence of any one pair is $(n_U|G|/|H|)/(|G|/|G_{xy}|)$. ■

If Γ contains a single nontrivial block system β, then the dual of the design $(\Omega, \Gamma, T, \phi)$ is defined simply by interchanging the roles of treatments and blocks. More precisely, if $\Delta = (\Omega, \{\mu, \beta, \varepsilon\}, T, \phi)$, then the dual Δ' of Δ is $(\Omega, \{\mu, \tau, \varepsilon\}, T', \phi')$, where T' is the set of β-blocks, τ is the partition of Ω such that each τ-block is equal to $\phi^{-1}(t)$ for some t in T, and the function $\phi' : \Omega \to T'$ is defined by

$$\phi'(\omega) = \text{the } \beta\text{-block containing } \omega, \qquad \text{for } \omega \text{ in } \Omega.$$

In [50] John proved that the dual of a thin generalized cyclic design (in the weak sense of [47]) is an ordinary cyclic design. The following theorem is a more general version of that result; John's result follows from the fact that quotient groups of cyclic groups are themselves cyclic.

THEOREM 4.5. *Let Δ be a G-design $(\Omega, \{\mu, \beta, \varepsilon\}, T, \phi)$, and let Δ' be the dual of Δ, with $\Delta' = (\Omega, \{\mu, \tau, \varepsilon\}, T', \phi')$. Then there exists a group G' of permutations of T' such that*

(i) *Δ' is a G'-design;*
(ii) *G' is abstractly isomorphic to a quotient group of G;*
(iii) *if Δ is a thin (G, γ)-design, then G' is transitive on T'.*

Proof. It is immediate from the definition of γ-morphism that a permutation g of Ω is a β-morphism of Δ if and only if it is a τ-morphism of Δ'. By Theorem 4.1 [(ii) $\Rightarrow$ (iii)], there is a faithful action of G on Ω as a group of β-morphisms of Δ. Let N be the intersection of the stabilizers in G of all the β-blocks. Then N is a normal subgroup of G, and the action of G on T' is a faithful action of the quotient group G/N. Put $G' = G/N$. Then Theorem 4.1 [(iii) $\Rightarrow$ (ii)] shows that Δ' is a (G', τ)-design. If Δ is a thin (G, β)-design, then G is transitive on T' and so G' is transitive on T'. ■

Note that, even if G is the largest permutation group on T such that Δ is a G-design, the group G' may not be the largest such group for T' and Δ'. For example, if p is the number of (G, β)-components in the homogeneous (G, β)-design Δ, then there is a transitive action of $G' \times \mathbf{S}_p$ on T' such that Δ' is a $(G' \times \mathbf{S}_p, \tau)$-design.

We conclude this section with a few remarks about the relationship between general balance and partial balance for G-designs. For each orbit U of G on $T \times T$, let A_U be its $T \times T$ *adjacency matrix*: the (t, u) entry of A_U is defined by

$$A_U(t,u) = \begin{cases} 1 & \text{if } (t,u) \in U, \\ 0 & \text{otherwise}. \end{cases}$$

These adjacency matrices span the centralizer algebra of G [17, Theorem II.1.3]. If they are all symmetric (a condition that has been termed *generous transitivity* of G by Neumann [64]), then the A_U form an association scheme. Hence Theorem 4.4 shows that each of the designs $(\Omega, \{\mu, \gamma, \varepsilon\}, T, \phi)$, for γ in Γ, is a partially balanced design in the sense of [24], and $(\Omega, \Gamma, T, \phi)$ is partially balanced in the extended sense of [41]. Hence $(\Omega, \Gamma, T, \phi)$ is generally balanced, the common eigenspaces of the concurrence matrices being the eigenspaces well known to be common to all partially balanced designs with that association scheme [22, 17, 13]. Shah [74] pointed out that, because concurrence matrices are symmetric, the conditions for an association scheme could be slightly weakened without destroying any of the essential features of partial balance. So long as G is cospectral, the matrices $A_U + A'_U$ do satisfy the weaker conditions of Shah, and such G-designs behave much like partially balanced designs.

5. THE PERMUTATION CHARACTER

In this section we return to our study of conditions on G which ensure that all G-central designs are generally balanced. Although, as we mentioned before, such conditions have been studied by many authors, to our knowledge no one has stated and proved exactly the result which we feel has the most practical significance for experimental design, so we shall do this in Theorems 5.4 and 5.5.

The statements of these results involve concepts from the theory of linear representations of a general finite permutation group. However, the special case in which G is a regular Abelian group is treated independently in Sections 6 and 7. Thus the reader whose interest is confined to generalized cyclic designs may omit this section, as none of the subsequent material, not even that in Section 6, depends on it in any way.

We need to consider linear representations of G over both the real and complex fields, $\mathbf{R}$ and $\mathbf{C}$, so we let $\mathbf{F}$ denote a field, which may be either $\mathbf{R}$ or $\mathbf{C}$.

DEFINITION. Suppose that there is a linear representation of the group G on a vector space V over $\mathbf{F}$, and that W is a subspace of V. Then W is *G-invariant* (with respect to this representation) if every element of G maps each vector in W to another vector in W. Further, W is *G-irreducible* if W is nontrivial and G-invariant but no nontrivial proper subspace of W is G-invariant. The linear representation is said to be *irreducible* if V is G-irreducible.

THEOREM 5.1. *If C is a real symmetric matrix in the centralizer algebra of the group G of permutations of the set T, then*

(i) $\mathbf{R}^T$ *is a direct sum of the eigenspaces of C and*
(ii) *every eigenspace of C is G-invariant.*

Proof. Since C is symmetric, (i) is a standard result of linear algebra. Let W be an eigenspace of C with eigenvalue λ. Let $w \in W$ and $g \in G$. Then

$$\begin{aligned}(wP_g)C &= wCP_g \qquad \text{(because } C \text{ commutes with } P_g\text{)}\\ &= (\lambda w)P_g\\ &= \lambda(wP_g)\end{aligned}$$

and so $wP_g \in W$. This proves (ii). ∎

Maschke's theorem [58, Section 1.6] shows that every eigenspace of the matrix C in Theorem 5.1 is a direct sum of G-irreducible subspaces. If we can show that $\mathbf{R}^T$ has a *unique* decomposition as a direct sum of G-irreducible subspaces, then each of these irreducible subspaces must be contained in an eigenspace of C for every concurrence matrix C: thus the set of concurrence matrices will be cospectral and the design will be generally balanced. The existence of a unique such decomposition can be decided by examining the *permutation character* of G, which is the function $\pi: G \to \mathbf{R}$ defined by

$$\pi(g) = |\{t \in T : t^g = t\}| \qquad \text{for } g \text{ in } G.$$

The characters of a finite group are the "calculus" of its linear representation theory, because they reduce many matrix problems to manageable calculations with complex numbers. Ledermann [58] provides a good introduction to the character theory of finite groups; however, none of the elementary texts includes all the results we need. Indeed, we have been unable to find our main result (Theorems 5.4 to 5.6) in any textbook; we shall therefore give a brief introduction to the necessary parts of character theory here.

DEFINITION. An **F**-character of the group G is the trace of a linear representation of G on a vector space over **F**: a character is thus a function from G to **F**. The character of an irreducible linear representation is said to be **F**-*irreducible*. An inner product $\langle\ ,\ \rangle$ is defined on characters by

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} .$$

It follows straight from the definition that, if χ is any **F**-character and 1 is the identity element of G, then $\chi(1)$ is the trace of the identity matrix and so is equal to the dimension of the corresponding vector space.

EXAMPLE 5.1.

1. The permutation character is the character of the permutation representation $g \mapsto P_g$; it can be thought of as an **R**-character or as a **C**-character.
2. For every field **F**, any group G has a *principal* character χ_0, defined by

$$\chi_0(g) = 1 \qquad \text{for all } g \text{ in } G;$$

it is always irreducible.
3. The **C**-irreducible characters of an Abelian group G are simply the group homomorphisms from G into the multiplicative group of **C**. Further details are given in Section 6.
4. If χ is a **C**-character of G, then so is $\bar{\chi}$, where $\bar{\chi}$ is defined by

$$\bar{\chi}(g) = \overline{\chi(g)} \qquad \text{for } g \text{ in } G.$$

Moreover, $\bar{\chi}$ is irreducible if and only if χ is.

The following fundamental result, due to Frobenius, establishes the importance of the irreducible characters (see [58, Sections 2.1–2.2]).

THEOREM 5.2. *For any finite group G, there are only a finite number of* **F**-*irreducible characters, and every* **F**-*character of G is the sum of these. These irreducible characters form an orthogonal basis, with respect to $\langle\ ,\ \rangle$, for the subspace of* $\mathbf{F}^G$ *spanned by all the* **F**-*characters of G. If* $\mathbf{F} = \mathbf{C}$, *then this basis is orthonormal.*

The $\mathbf{C}$-irreducible characters of a group are usually displayed in a *character table*, such as Tables 4 and 5 below (see [58, Section 2.3]). The practical utility of the results in this section depends on knowing the character table of G. Now, the calculation of the character table of an arbitrary group can be an extremely lengthy computation. However, those groups G for which G-central designs are used or proposed appear largely to be groups whose character tables either are already known or can be easily calculated from those which are known.

Let $I_{\mathbf{F}}$ be the set of $\mathbf{F}$-irreducible characters of G. Since we may regard the permutation linear representation of G as being on either $\mathbf{R}^T$ or $\mathbf{C}^T$, Theorem 5.2 shows that

$$\pi = \sum_{\chi \in I_{\mathbf{F}}} n_\chi^{\mathbf{F}} \chi, \qquad \text{where} \quad n_\chi^{\mathbf{F}} = \frac{\langle \pi, \chi \rangle}{\langle \chi, \chi \rangle}.$$

The number $n_\chi^{\mathbf{F}}$ is a nonnegative integer called the *multiplicity* of χ in π. The permutation character is said to be $\mathbf{F}$-*multiplicity-free* if $n_\chi^{\mathbf{F}} \in \{0, 1\}$ for all χ in $I_{\mathbf{F}}$; that is, no irreducible character appears more than once in the sum.

Serre [73, Section 2.6] gives an important decomposition of $\mathbf{C}^T$. A slight modification of his Proposition 6 gives a result in a form applicable to both $\mathbf{C}$ and $\mathbf{R}$, which we now state. (In [5], Andersson also states part of this theorem with $\mathbf{F} = \mathbf{R}$.)

THEOREM 5.3. *There are orthogonal G-invariant subspaces $(V_\chi)_{\chi \in I_{\mathbf{F}}}$ of $\mathbf{F}^T$ called the G-homogeneous subspaces of $\mathbf{F}^T$, such that:*

(i) $\mathbf{F}^T = \bigoplus_{\chi \in I_{\mathbf{F}}} V_\chi$ *(this decomposition is called the G-homogeneous decomposition of $\mathbf{F}^T$);*

(ii) $\dim(V_\chi) = n_\chi^{\mathbf{F}} \chi(1)$, *where 1 is the identity element of G;*

(iii) *if W is any G-irreducible subspace of $\mathbf{F}^T$, then there is some χ in $I_{\mathbf{F}}$ such that $W \subseteq V_\chi$, the character of the restriction to W of the permutation linear representation of G is χ, and therefore* $\dim(W) = \chi(1)$;

(iv) *the matrix of orthogonal projection onto V_χ is*

$$\frac{\chi(1)}{|G|\langle \chi, \chi \rangle} \sum_{g \in G} \bar{\chi}(g) P_g;$$

(v) *the subspace V_χ is G-irreducible if and only if $n_\chi^{\mathbf{F}} = 1$; if $n_\chi^{\mathbf{F}} \geqslant 2$, then there is no unique decomposition of V_χ into G-irreducible subspaces.*

Since the concurrence matrices are *real* symmetric matrices, we need to describe the G-homogeneous decomposition of $\mathbf{R}^T$ in the case when G is cospectral. The results we need are summarized in the following portmanteau theorem.

THEOREM 5.4. *Let G be a group of permutations of a set T. Then the following conditions are all equivalent*:

(i) *G is cospectral*;

(ii) *the set of symmetric matrices in the real centralizer algebra of G is commutative*;

(iii) *the set of diagonalizable matrices in the real centralizer algebra of G is commutative*;

(iv) *the permutation character π is $\mathbf{R}$-multiplicity-free*;

(v) *the G-homogeneous subspaces of $\mathbf{R}^T$ are G-irreducible*;

(vi) *$\mathbf{R}^T$ has a unique decomposition as a direct sum of G-irreducible subspaces*;

(vii) *every G-homogeneous subspace of $\mathbf{R}^T$ is contained in an eigenspace of every diagonalizable matrix in the real centralizer algebra of G*;

(viii) *the set of diagonalizable matrices in the real centralizer algebra of G is cospectral.*

Proof. Theorem 5.3 shows that conditions (iv), (v), and (vi) are equivalent. By Lemma 2.1, conditions (i) and (ii) are equivalent, as are conditions (iii) and (viii). We shall prove that (ii) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (vi): Let W be a G-invariant subspace of $\mathbf{R}^T$, and let Q be the matrix of orthogonal projection onto W. It is straightforward to check that Q is in the centralizer algebra of G.

If condition (vi) is false, then there are G-irreducible subspaces W_1, W_2 of $\mathbf{R}^T$ such that $W_1 \cap W_2 = \{0\}$ but W_1 and W_2 are not orthogonal to each other. Let Q_1 and Q_2 be the matrices of orthogonal projection onto W_1 and W_2 respectively. Then Q_1 and Q_2 are symmetric matrices in the real centralizer algebra of G, but [37, Section 76] shows that Q_1 and Q_2 do not commute with each other. Thus condition (ii) is false.

(vi) $\Rightarrow$ (vii): Let C be a diagonalizable matrix in the real centralizer algebra of G. Then $\mathbf{R}^T$ is a direct sum of the eigenspaces of C, each of which is G-invariant, by Theorem 5.1. This decomposition of $\mathbf{R}^T$must therefore be coarser than the unique decomposition into G-irreducibles, whose components are precisely the G-homogeneous subspaces. Therefore every G-homogeneous subspace is contained in an eigenspace of C.

(vii) $\Rightarrow$ (viii): Obvious.

(viii) $\Rightarrow$ (ii): Symmetric matrices are diagonalizable, so (viii) implies (i), which is equivalent to (ii). ■

Theorem 5.4 gives us conditions equivalent to G being cospectral. In conjunction with Theorem 5.3, it also gives formulas for the relevant projectors and dimensions in terms of the **R**-irreducible characters. However, we feel that criteria and formulas couched in terms of the **R**-irreducible characters of G are not very useful in practice, since character tables almost always show the **C**-irreducible characters of G; this is partly because the latter can be easily identified as being those characters χ for which $\langle \chi, \chi \rangle = 1$.

Thus we shall modify Theorem 5.3 to describe a G-invariant decomposition of $\mathbf{R}^T$ in terms of the **C**-irreducible characters of G and show that if G is cospectral, then all G-central designs are generally balanced with respect to this decomposition. This can be done using the following important classification of **C**-irreducible characters into three *types* (see [73, Sections 12.1, 12.2, 13.4]).

A **C**-irreducible character of G is:

complex if there is an element g of G for which $\chi(g)$ is not real;

real if it is realizable as the character of a linear representation of G on a real vector space;

quaternionic otherwise.

The *type* of a character can in fact be readily calculated from the character table of G, because

$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0 & \text{if} \quad \chi \text{ is complex,} \\ +1 & \text{if} \quad \chi \text{ is real,} \\ -1 & \text{if} \quad \chi \text{ is quaternionic.} \end{cases}$$

The set of complex **C**-irreducible characters is a disjoint union of two equinumerous sets I_c and $I_{c'}$, such that $\chi \in I_c$ if and only if $\bar{\chi} \in I_{c'}$. Let I_r and I_q be the sets of real and quaternionic **C**-irreducible characters respectively. Serre [73, Section 3.2] shows that there is a bijection

$$f\colon I_c \cup I_r \cup I_q \to I_{\mathbf{R}}$$

given by

$$f(\chi) = \begin{cases} \chi + \bar{\chi} & \text{if} \quad \chi \in I_c, \\ \chi & \text{if} \quad \chi \in I_r, \\ 2\chi & \text{if} \quad \chi \in I_q. \end{cases}$$

Moreover, since π is the character of a real representation,

$$n^{\mathbf{R}}_{f(\chi)} = \begin{cases} n^{\mathbf{C}}_{\chi} & \text{if} \quad \chi \in I_c \cup I_r, \\ \frac{1}{2} n^{\mathbf{C}}_{\chi} & \text{if} \quad \chi \in I_q. \end{cases}$$

Let us say that the **C**-irreducible character χ of G is *π-indecomposable* if $n^{\mathbf{R}}_{f(\chi)} \in \{0, 1\}$.

The formulas in the following theorem now follow directly from those in Theorem 5.3.

THEOREM 5.5.

(i) *The G-homogeneous subspaces of $\mathbf{R}^T$ are W_χ for χ in $I_c \cup I_r \cup I_q$, where the matrix Q_χ of orthogonal projection onto W_χ is given by*

$$Q_\chi = \begin{cases} \dfrac{\chi(1)}{|G|} \sum_{g \in G} [\chi(g) + \bar{\chi}(g)] P_g & \textit{if} \quad \chi \in I_c, \\ \dfrac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) P_g & \textit{if} \quad \chi \in I_r \cup I_q. \end{cases}$$

Moreover,

$$\dim(W_\chi) = \begin{cases} 2 n^{\mathbf{C}}_{\chi} \chi(1) & \textit{if} \quad \chi \in I_c, \\ n^{\mathbf{C}}_{\chi} \chi(1) & \textit{if} \quad \chi \in I_r \cup I_q. \end{cases}$$

(ii) *If W is any G-irreducible subspace of $\mathbf{R}^T$, then there is some χ in $I_c \cup I_r \cup I_q$ such that $W \subseteq W_\chi$ and $\dim(W) = d_\chi \chi(1)$, where*

$$d_\chi = \begin{cases} 1 & \textit{if} \quad \chi \in I_r \\ 2 & \textit{if} \quad \chi \in I_c \cup I_q. \end{cases}$$

We have now introduced all the terminology required to state the following widely applicable theorem, whose proof follows directly from Theorems 5.4 and 5.5.

THEOREM 5.6. *The permutation group G is cospectral if and only if every **C**-irreducible character of G is π-indecomposable, where π is the*

TABLE 4

CHARACTER TABLE OF $\mathbf{S}_3$

	1	a, a^2	b, ab, a^2b
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0
π	6	0	0

permutation character of G. Moreover, if G is cospectral, then all G-central designs are generally balanced with respect to the G-homogeneous decomposition of $\mathbf{R}^T$.

McLaren stated part of Theorem 5.6 in [60]. He defined the set of real symmetric matrices in the centralizer algebra of G to be *properly constrained by G* if the permutation character π of G is $\mathbf{R}$-multiplicity-free.

EXAMPLE 5.2 (Example 3.1 revisited). Let T and G be as in Example 3.1. The character table of $\mathbf{S}_3$ is shown in Table 4: the permutation character π of G can be decomposed as $\pi = \chi_0 + \chi_1 + 2\chi_2$. Since χ_2 is a real character, this shows that G is not cospectral, and so G-central designs may not be generally balanced, as Example 3.1 shows.

EXAMPLE 5.3. Suppose that the treatments consist of the ten genotypes of some plant obtained by crossing all pairs of five pure parental lines, but omitting self-crosses and ignoring the gender of the parents. Then we may identify T with the set of unordered pairs from $\{1,2,3,4,5\}$. Suppose further that there are two plants per block, there being one block for each pair of genotypes with no parental lines in common. Then $|\Omega| = 30$, and the treatment concurrence graph [66, 65] is the Petersen graph (see [39, Chapter 9] or [69]) shown in Figure 1. Let G be the symmetric group $\mathbf{S}_5$ in its action on unordered pairs, and let π be the corresponding permutation character. A fragment of the character table of $\mathbf{S}_5$ is shown in Table 5 (using the usual notation for the conjugacy classes, which is given in [58]): the decomposition of π is $\pi = \chi_0 + \chi_1 + \chi_2$.

For i in $\{1,2,3,4,5\}$, define the element v_i of $\mathbf{R}^T$ by

$$tv_i = \begin{cases} 1 & \text{if} \quad i \in t, \\ 0 & \text{otherwise}. \end{cases}$$

Let W_0 and W be the subspaces of $\mathbf{R}^T$ spanned by $v_1 + v_2 + \cdots + v_5$ and

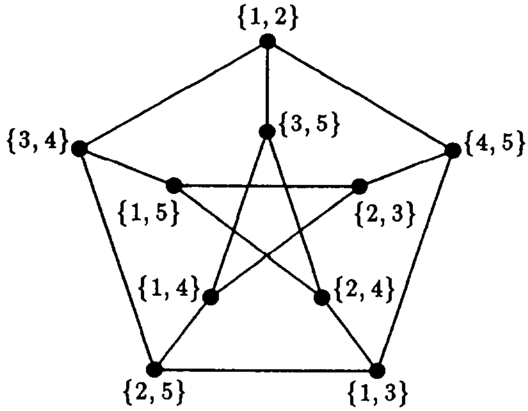

FIG. 1. The Petersen graph.

$\{v_1, v_2, v_3, v_4, v_5\}$ respectively. Then it can be shown that the characters χ_0, χ_1, χ_2 are all real and that the corresponding $\mathbf{S}_5$-homogeneous subspaces of $\mathbf{R}^T$ are W_0, W_1, W_2, where $W_1 = W \cap W_0^{\perp}$ and $W_2 = W^{\perp}$. The decomposition $W_0 \oplus W_1 \oplus W_2$ is very strongly related to the natural structure of T. Since π is $\mathbf{R}$-multiplicity-free, the design is generally balanced with respect to this decomposition.

This example may be generalized: whatever the number of pure parental lines, any block design for which the concurrence of two genotypes depends only on how many parental lines they have in common is generally balanced with respect to a decomposition analogous to the one above. In [76, Section 2.3(b)(ii)], Sinha has observed that all such designs are simple in his sense.

Even if G is not cospectral, the analysis of the permutation character in terms of the $\mathbf{C}$-irreducible characters of G is still useful. A design may be

TABLE 5
FRAGMENT OF THE CHARACTER TABLE OF $\mathbf{S}_5$

	1	2	2^2	3	$2\cdot 3$	4	5
χ_0	1	1	1	1	1	1	1
χ_1	4	2	0	1	-1	0	-1
χ_2	5	1	1	-1	1	-1	0
χ_3	6	0	-2	0	0	0	1
$\vdots$							
π	10	4	2	1	1	0	0
π'	20	6	0	2	0	0	0

known to be generally balanced from other considerations (for example, an ordinary block design). Then the common eigenspaces of the concurrence matrices are direct sums of G-irreducible subspaces. Part (ii) of Theorem 5.5 shows that these G-irreducible subspaces may be found by examining each G-homogeneous subspace separately, rather than by examining the whole of $\mathbf{R}^T$, and shows what dimensions these G-irreducible subspaces have. In particular, if χ is π-indecomposable, then W_χ is itself contained in the common eigenspaces of the concurrences matrices. Thus Theorem 5.5 can still help in the task of finding a decomposition of $\mathbf{R}^T$ with respect to which the design is generally balanced, even though this decomposition may not be entirely determined by G.

EXAMPLE 5.4. Let T and G be as in Example 3.1. Let W_0, W_1, W_2 be the G-homogeneous subspaces of $\mathbf{R}^T$ corresponding to χ_0, χ_1, χ_2 respectively: then W_0 is spanned by $(1,1,1,1,1,1)$, W_1 is spanned by $(1,1,1,-1,-1,-1)$, and $W_2=(W_0 \oplus W_1)^\perp$.

Consider the class of all ordinary block designs which are G-central. Since each of these designs has a single nontrivial block system, they are all generally balanced. The irreducible characters χ_0 and χ_1 are π-indecomposable, so the G-homogeneous spaces W_0 and W_1 are eigenspaces of the concurrence matrices of all these designs. However, the space W_2 is, in general, a direct sum of two 2-dimensional eigenspaces, and these summands are, in general, different for different designs.

In particular, consider the designs Δ_1 and Δ_2, where Δ_1 consists of the β-blocks in the right-hand column of Table 1, and Δ_2 is the design in Table 6, in six blocks of size 2 (this design was given, in a different form, in [74]). The design Δ_1 is generally balanced with respect to the decomposition

$$W_0 \oplus W_1 \oplus W_3 \oplus W_4,$$

and Δ_2 is generally balanced with respect to

$$W_0 \oplus W_1 \oplus W_3' \oplus W_4',$$

where W_3 is spanned by $(1,-1,0,1,0,-1)$ and $(1,1,-2,1,-2,1)$, $W_4=W_2 \cap W_3^\perp$, W_3' is spanned by $(2,-1,-1,-\sqrt{3},0,\sqrt{3})$ and $(0,\sqrt{3},-\sqrt{3},-1,2,-1)$,

TABLE 6

DESIGN Δ_2 IN EXAMPLE 5.4

[1	b]	[a	a^2b]	[a^2	ab]
[1	b]	[a	a^2b]	[a^2	ab]
[1	ab]	[a	b]	[a^2	a^2b]

and $W_4' = W_2 \cap W_3'^{\perp}$. There is no decomposition of $\mathbf{R}^T$ with respect to which *both* Δ_1 and Δ_2 are generally balanced.

EXAMPLE 5.5. It is instructive to compare Example 5.3 with a diallel example like those of [45], where the parental genders are distinguished. Thus $|T| = 20$ in this case. The permutation character π' of $\mathbf{S}_5$ in its action on ordered pairs (omitting self-pairs) has decomposition

$$\pi' = \chi_0 + 2\chi_1 + \chi_2 + \chi_3$$

(these irreducible characters are shown in Table 5).

For i, j in $\{1,2,3,4,5\}$ define the vectors $x_{ij}, y_{ij}, z_{ij}, v_i, w_i$ by

$$tx_{ij} = \begin{cases} 1 & \text{if} \quad t = (i,j), \\ 0 & \text{otherwise}; \end{cases}$$

$$y_{ij} = x_{ij} + x_{ji};$$

$$z_{ij} = x_{ij} - x_{ji};$$

$$v_i = \sum_j x_{ij};$$

$$w_i = \sum_j x_{ji}.$$

Let Y, Z, V, W be the subspaces of $\mathbf{R}^T$ spanned by

$$\{y_{ij}: 1 \leqslant i, j \leqslant 5\}, \{z_{ij}: 1 \leqslant i, j \leqslant 5\}, \{v_i: 1 \leqslant i \leqslant 5\}, \text{and } \{w_i: 1 \leqslant i \leqslant 5\}.$$

It can be shown that the $\mathbf{S}_5$-homogeneous subspaces of $\mathbf{R}^T$ corresponding to $\chi_0, \chi_1, \chi_2, \chi_3$ are W_0, W_1, W_2, W_3 respectively, where W_0 consists of the constant vectors, $W_1 = (V+W) \cap W_0^{\perp}$, $W_2 = Y \cap (V+W)^{\perp}$, and $W_3 = Z \cap (V+W)^{\perp}$. The spaces W_0, W_2, W_3 are common eigenspaces of the concurrence matrices of every $\mathbf{S}_5$-central design with treatment set T; in general, W_1 is a sum of two 4-dimensional eigenspaces. The two spaces W_2 and W_3 have a natural interpretation, because W_2 consists of all contrasts which are symmetric in the parental genders and orthogonal to each parent, while W_3 consists of all contrasts which are antisymmetric in the parental genders and orthogonal to each parent.

If G is not transitive, then the multiplicity of the principal character χ_0 is greater than 1. Since the principal character is real, this shows that G is not cospectral. Recall that, for an Abelian group, transitivity is equivalent to regularity. [58, Sections 2.2 and 2.4] shows that, if G is a regular Abelian group, then $n_\chi^{\mathbf{C}} = 1$ for all χ in $I_{\mathbf{C}}$. Hence we obtain the following theorem, which will also be proved in Section 7 independently of the results of the present section.

THEOREM 5.7. *If G is a regular Abelian group, then all G-central designs are generally balanced with respect to the G-homogeneous decomposition of* $\mathbf{R}^T$.

6. CHARACTERS OF ABELIAN GROUPS

In this section we introduce some results concerning the irreducible characters of an Abelian group and their relationship to the regular representation of the group; it is, however, completely independent of Section 5. These results will be used in Section 7 to prove an important result: that, if G is an Abelian group, then any G-design is generally balanced. So, from now on, G will always be a finite Abelian group acting regularly on T, which will be identified with G; the resulting G-design is called an *Abelian group design*.

An irreducible character χ of G is a nonzero map $\chi: G \to \mathbf{C}$ such that $\chi(g)\chi(h) = \chi(gh)$ for all g, h in G. Under the pointwise multiplication defined by

$$(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g) \qquad \text{for all } g \text{ in } G,$$

the irreducible characters form a group, the *dual group* G^* of G, and G^* is isomorphic to G. The identity element of G^* is the *principal character* χ_0 of G defined by $\chi_0(g) = 1$ for all g in G. For all χ in G^*, the inverse of χ is the irreducible character $\bar{\chi}$ such that $\bar{\chi}(g) = \overline{\chi(g)}$ for all g in G. (See [42, Section V.6] or [58, Section 2.4].) The following hold for all irreducible characters χ of G:

$$\chi(1_G) = 1, \qquad \text{where } 1_G \text{ is the identity element of } G; \tag{4}$$

$$\bar{\chi}(g) = \chi(g^{-1}) \qquad \text{for all } g \text{ in } G; \tag{5}$$

$$\frac{1}{|G|}\sum_{g \in G} \chi(g) = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Since G is finite, there is a smallest positive integer e, called the *exponent* of G, such that $g^e = 1_G$ for all g in G. Thus, for any χ in G^*, we have $\chi(g)^e = \chi(g^e) = \chi(1_G) = 1$, so $\chi(G) \subseteq \{\epsilon^i : i \in \mathbf{N}\}$, where ϵ is a primitive eth root of 1 in $\mathbf{C}$. Thus $\chi = \bar{\chi}$ if and only if $\chi(g) = \pm 1$ for all g in G.

If we put $G_r^* = \{\chi \in G^* : \chi = \bar{\chi}\}$, then we may partition $G^* \setminus G_r^*$ into two equinumerous sets G_c^*, $G_{c'}^*$ such that $\chi \in G_c^*$ if and only if $\bar{\chi} \in G_{c'}^*$.

THEOREM 6.1. *For χ in G^* and h in G, let $v(\chi, h)$ be the vector in $\mathbf{R}^G$ whose gth coordinate is $\chi(hg) + \bar{\chi}(hg)$. Let W_χ be the subspace of $\mathbf{R}^G$ spanned by the set of vectors $\{v(\chi, h) : h \in G\}$. Then*

(i) *if χ and ψ are irreducible characters, then $W_\chi = W_\psi$ if and only if $\psi \in \{\chi, \bar{\chi}\}$;*

(ii) *if χ is an irreducible character, then*

$$\dim(W_\chi) = \begin{cases} 1 & \text{if } \chi = \bar{\chi} \\ 2 & \text{otherwise;} \end{cases}$$

(iii) *the vector space $\mathbf{R}^G$ is the orthogonal direct sum of the subspaces W_χ for $\chi \in G_r^* \cup G_c^*$.*

Proof. See [57]. ∎

The decomposition of $\mathbf{R}^G$ in part (iii) of Theorem 6.1 is called the *G-homogeneous decomposition* of $\mathbf{R}^G$, and the subspaces $(W_\chi)_{\chi \in G^*}$ are called the *G-homogeneous* subspaces of $\mathbf{R}^G$. Those who have read Section 5 should note that there is no conflict of terminology here: using Theorem 5.5 and the fact that, for χ, ψ in G^* and h in G,

$$\frac{1}{|G|} \sum_{g \in G} v(\chi, h)[\psi(g) + \bar{\psi}(g)] P_g$$

$$= \begin{cases} v(\chi, h) & \text{if } \psi \neq \bar{\psi} \text{ and } \chi \in \{\psi, \bar{\psi}\}, \\ 2v(\chi, h) & \text{if } \chi = \psi = \bar{\psi}, \\ 0 & \text{otherwise,} \end{cases}$$

it can be shown that the spaces W_χ defined in Theorems 5.5 and 6.1 are identical. Moreover, Theorem 5.4 and the remarks preceding Theorem 5.7 show that each G-homogeneous subspace W_χ is G-irreducible.

The irreducible characters of the group have been used to decompose the vector space $\mathbf{R}^G$ of an Abelian group design in other work [27, 15, 10, 56, 57], but the resulting decompositions are not identical. The G-homogeneous decomposition is the finest that is possible, since the components are G-irreducible, but other useful decompositions may be found by combining appropriate subspaces W_χ; for example, the decompositions in [10] and [15] both use the subspaces W_H defined as follows: for each cyclic subgroup H of G^*, the subspace W_H of $\mathbf{R}^G$ is the sum of the subspaces W_χ for those χ such that $\langle\chi\rangle = H$ (that is, W_H is obtained by combining all the subspaces corresponding to generators of the cyclic subgroup H). It is straightforward to check that this decomposition is identical to the G-homogeneous decomposition if and only if the exponent e of G divides 12.

7. ABELIAN GROUP DESIGNS

THEOREM 7.1. *Every Abelian group design is generally balanced with respect to its homogeneous decomposition (given in Theorem 6.1).*

Although Theorem 7.1 was proved as Theorem 5.7 using the general theory, we shall now give an alternative proof by finding an explicit formula for the eigenvalue of the concurrence matrix C_γ $(\gamma \in \Gamma)$ on the homogeneous subspace W_χ $(\chi \in G^*)$.

The proof of Theorem 4.2 shows that the concurrence of treatments t and u in γ-blocks depends only on the orbit of G on $G \times G$ which contains (t, u). Because G is Abelian, these orbits are indexed by the elements of G, and are of the form U_g, where $U_g = \{(t, u) : tu^{-1} = g\}$. The adjacency matrix of the orbit U_g is the permutation matrix $P_{g^{-1}}$. We can therefore define the following notation. For g in G, let $c_{g\gamma}$ be the common concurrence in γ-blocks of treatment pairs in the orbit U_g. Thus $C_\gamma = \sum_{g \in G} c_{g\gamma} P_{g^{-1}}$. Each occurrence in a γ-block of a pair (t, u) with $tu^{-1} = g$ corresponds to an occurrence in the same γ-block of the pair (u, t) with $ut^{-1} = g^{-1}$. Thus $c_{g^{-1}\gamma} = c_{g\gamma}$ for all g in G. For χ in G^* and γ in Γ, define

$$\nu_{\gamma\chi} = \frac{1}{rk_\gamma} \sum_{g \in G} c_{g\gamma} \chi(g). \tag{7}$$

Thus

$$\nu_{\gamma\bar{\chi}} = \frac{1}{rk_\gamma} \sum_{g \in G} c_{g\gamma} \bar{\chi}(g) = \frac{1}{rk_\gamma} \sum_{g \in G} c_{g^{-1}\gamma} \chi(g^{-1}), \qquad \text{by Equation (5).}$$

Since summation over $g \in G$ is the same as summation over $g^{-1} \in G$, this implies that $\nu_{\gamma\chi} = \nu_{\gamma\bar{\chi}}$. A similar argument shows that $\chi(g)$ can be replaced by the real number $\frac{1}{2}[\chi(g) + \bar{\chi}(g)]$ in the definition of $\nu_{\gamma\chi}$.

THEOREM 7.2. *Let G be an Abelian group, and let $(\Omega, \Gamma, G, \phi)$ be a G-design. Then, for all γ in Γ and all χ in G^*, the G-homogeneous* *space W_χ is a subspace of an eigenspace of the concurrence matrix C_γ with eigenvalue $rk_\gamma \nu_{\gamma\chi}$.*

Proof. We have

$$C_\gamma = \sum_{g \in G} c_{g\gamma} P_{g^{-1}} = \sum_{g \in G} \tfrac{1}{2} c_{g\gamma}\left(P_g + P_{g^{-1}}\right), \qquad \text{since} \quad c_{g\gamma} = c_{g^{-1}\gamma}.$$

For χ in G^* and h in G, let v be the vector $v(\chi, h)$ in $\mathbf{R}^G$ specified in Theorem 6.1. Thus, for all t in G,

$$tv = \chi(ht) + \bar{\chi}(ht).$$

(Recall that our conventions of Section 3 require us to write vectors and matrices on the *right* of their arguments: however, it is convenient to continue to write characters on the *left* of their arguments.) By Equation (3),

$$\begin{aligned}
t\left[v\left(P_g + P_{g^{-1}}\right)\right] &= (t^g)v + \left(t^{g^{-1}}\right)v \\
&= (tg)v + \left(tg^{-1}\right)v \\
&= \chi(htg) + \bar{\chi}(htg) + \chi\left(htg^{-1}\right) + \bar{\chi}\left(htg^{-1}\right) \\
&= \left[\chi(ht) + \bar{\chi}(ht)\right]\left[\chi(g) + \chi(g^{-1})\right] \qquad \text{[by Equation (5)]} \\
&= (tv)\left[\chi(g) + \chi(g^{-1})\right].
\end{aligned}$$

Thus $v(P_g + P_{g^{-1}}) = v[\chi(g) + \chi(g^{-1})]$. Since $\chi(g) + \chi(g^{-1})$ is a scalar, this shows that v is an eigenvector of $P_g + P_{g^{-1}}$ with eigenvalue $\chi(g) + \chi(g^{-1})$. Hence

$$\begin{aligned}
vC_\gamma &= v \sum_{g \in G} \tfrac{1}{2} c_{g\gamma}\left(P_g + P_{g^{-1}}\right) \\
&= \sum_{g \in G} \tfrac{1}{2} c_{g\gamma} v\left(P_g + P_{g^{-1}}\right) \\
&= \left[\sum_{g \in G} \tfrac{1}{2} c_{g\gamma}\left[\chi(g) + \chi(g^{-1})\right]\right] v \\
&= \left[\sum_{g \in G} c_{g\gamma}\chi(g)\right] v \qquad \left(\text{since} \quad c_{g\gamma} = c_{g^{-1}\gamma}\right) \\
&= rk_\gamma \nu_{\gamma\chi} v.
\end{aligned}$$

■

This completes the proof of Theorem 7.1. The numbers $\nu_{\gamma\chi}$ and the block structure of the design can be used to calculate its efficiency factors, as we now show.

THEOREM 7.3. *Let $(\Omega, \Gamma, G, \phi)$ be an Abelian group design. For all α in Γ and all χ in G^*, the efficiency factor $\lambda_{\alpha\chi}$ for the G-homogeneous treatment subspace W_χ in stratum $\mathscr{S}_\alpha$ is given by*

$$\lambda_{\alpha\chi} = \sum_{\gamma\in\Gamma} m(\alpha,\gamma)\nu_{\gamma\chi},$$

where m is the Möbius function of Γ and $\nu_{\gamma\chi}$ is defined by Equation (7).

Proof. The efficiency factor $\lambda_{\alpha\chi}$ is equal to the eigenvalue of $r^{-1}L_\alpha$ on W_χ. Now

$$\begin{aligned} L_\alpha &= X'S_\alpha X \\ &= \sum_{\gamma\in\Gamma} m(\alpha,\gamma)X'B_\gamma X \qquad \text{[by Equation (2)]} \\ &= \sum_{\gamma\in\Gamma} \frac{m(\alpha,\gamma)}{k_\gamma} C_\gamma. \end{aligned}$$

The result therefore follows from Theorem 7.2. ■

The usefulness of Theorem 7.3 depends on our ability to calculate the concurrences $c_{g\gamma}$. It turns out that this calculation is particularly easy for thin Abelian group designs (see Section 4 for the definition of thin designs). However, it is impossible for a design to be thin with respect to every γ in Γ. Thus it is convenient to do the calculation for each γ in Γ separately by decomposing the design into thin (G,γ)-designs. So we now fix γ in Γ.

Suppose that Δ is a G-design, where $\Delta = (\Omega, \Gamma, G, \phi)$. Then, for some set J, there is a partition $(\Omega_j)_{j\in J}$ of Ω into (G,γ)-components. For each j in J, let b_j be any γ-block in Ω_j and let s_j be the order of the stabilizer in G of the multiset of b_j. For j in J and g in G define

$$n_{jg} = \left|\left\{(\omega,\theta)\in b_j\times b_j : \phi(\omega)[\phi(\theta)]^{-1} = g\right\}\right|.$$

LEMMA 7.4. *With the above notation,*

$$\sum_{j \in J} \frac{1}{s_j} = \frac{r}{k_\gamma}.$$

In particular, if s_j is constant over j in J, then it is equal to $k_\gamma |J|/r$.

Proof. The orbit-stabilizer theorem [81, Theorem 3.2] shows that the number of γ-blocks in Ω_j is equal to $|G|/s_j$. Thus

$$|G| \times r = |\Omega| = \left(\sum_{j \in J} \frac{|G|}{s_j} \right) \times k_\gamma. \qquad \blacksquare$$

THEOREM 7.5. *For all g in G, the concurrence $c_{g\gamma}$ is equal to $\sum_{j \in J} n_{jg}/s_j$. In particular, if s_j is constant over j in J, then $c_{g\gamma} = (r/k_\gamma |J|) \sum_{j \in J} n_{jg}$, while if $n_{jg} = n_g$ for all j in J, then $c_{g\gamma} = rn_g/k_\gamma$.*

Proof. Fix j in J. Let γ_j, ε_j be the partitions of Ω_j into γ-blocks and singletons respectively, and put $\Delta_j = (\Omega, \{\gamma_j, \varepsilon_j\}, G, \phi)$. Then Δ_j is a thin (G, γ_j)-design. Let U be any orbit of G on $G \times G$; if $U = U_g$, then for the design Δ_j the integer n_U defined in Theorem 4.4 will equal n_{jg}. Since G acts regularly on itself, $G_{tu} = \{1_G\}$ for all t, u in G. Theorem 4.4 shows that if $(t, u) \in U_g$ then the concurrence of t and u in γ_j-blocks in Δ_j is n_{jg}/s_j. Hence

$$c_{g\gamma} = \sum_{j \in J} \frac{n_{jg}}{s_j}. \qquad \blacksquare$$

The numbers n_{jg} may be counted directly from the "table of differences" [18, Chapter 6]. If s_j is not known *a priori*, it may be found by generating all γ-blocks in Δ_j, since the number of these is equal to $|G|/s_j$. However, if Δ is homogeneous, the formula for $c_{g\gamma}$, and hence $\nu_{\gamma\chi}$, does not involve s_j.

COROLLARY 7.6. *If Δ is a homogeneous (G, γ)-design, then*

$$\nu_{\gamma\chi} = \frac{1}{k_\gamma^2} \sum_{g \in G} n_g \chi(g),$$

where n_g is calculated from any γ-block of Δ.

Suppose that the design Δ is binary with respect to γ. Then the trace of C_γ is equal to $|\Omega|$. Since the trace of a matrix is the sum of its eigenvalues, taking account of multiplicities, Theorem 7.2 shows that

$$\sum_{\chi \in G^*} \nu_{\gamma\chi} = \frac{|\Omega|}{rk_\gamma},$$

and this is equal to the number of γ-blocks divided by the replication number. This formula provides a useful check when the values $\nu_{\gamma\chi}$ are calculated.

The trivial concurrence matrices may, of course, be written down directly. If $g \neq 1$ then $c_{g\varepsilon} = 0$, while $c_{1\varepsilon} = r$. Thus, for all χ in G^*,

$$\nu_{\varepsilon\chi} = \chi(1_G) = 1$$

by Equation (4). On the other hand, $c_{g\mu} = r^2$ for all g in G, and $k_\mu = r|G|$, so

$$\begin{aligned} \nu_{\mu\chi} &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \\ &= \begin{cases} 1 & \text{if } \chi \text{ is the principal character,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

by Equation (6).

EXAMPLE 7.1. Suppose that Ω consists of 54 plots with the following block structure. There are three σ-blocks, each of size 18, and each σ-block consists of, on the one hand, three ρ-blocks of size 6 and, on the other, three κ-blocks of size 6. Within a σ-block, ρ-blocks and κ-blocks are completely crossed, but the block system formed by the intersections of ρ- and κ-blocks is *not* in this block structure; thus if $\Gamma = \{\mu, \sigma, \rho, \kappa, \varepsilon\}$, then (Ω, Γ) is a Tjur block structure which is *not* an orthogonal block structure (in the sense of [14]).

Let G be the cyclic group of order 9, written additively, and let Δ be the design generated by G from the initial σ-block shown in Table 7. This design is homogeneous for every γ in Γ, so we shall use Corollary 7.6 to calculate the $\nu_{\gamma\chi}$. This requires the values of n_g for g in G, which are easily calculated using the information in Table 7. For the nontrivial block systems this is done by calculating a table of differences from any block of that system, and the results are given in Table 8.

TABLE 7

INITIAL σ-BLOCK FOR THE DESIGN IN EXAMPLE 7.1[a]

0	1	3	4	6	7
2	7	5	1	8	4
3	5	6	8	0	2

[a] Each row is a ρ-block, and each pair of columns is a κ-block.

TABLE 8

VALUES OF n_g FOR THE DESIGN IN EXAMPLE 7.1

Block system	n_g				
	$g=0$	± 1	± 2	± 3	± 4
σ	36	36	36	36	36
ρ	6	3	3	6	3
κ	6	3	5	3	4

The irreducible characters of G are $\{\chi_h : h \in G\}$, where

$$\chi_h(g) = \epsilon^{gh}, \qquad \epsilon \text{ being a primitive 9th root of 1.}$$

The values of $\nu_{\gamma\chi}$ are shown in Table 9. For brevity, we write e_i for the real number $\epsilon^i + \epsilon^{-i}$ for $i \in \{1,2,4\}$.

TABLE 9

VALUES OF $\nu_{\gamma\chi}$ FOR THE DESIGN IN EXAMPLE 7.1

γ	χ_0	χ_1	χ_2	χ_3	χ_4
μ	1	0	0	0	0
σ	1	0	0	0	0
ρ	1	0	0	$\frac{1}{4}$	0
κ	1	$\dfrac{3+e_2-e_1}{36}$	$\dfrac{3+e_4-e_2}{36}$	0	$\dfrac{3+e_1-e_4}{36}$
ε	1	1	1	1	1

$$e_1 = \epsilon^1 + \epsilon^8 = 2\cos(2\pi/9)$$
$$e_2 = \epsilon^2 + \epsilon^7 = 2\cos(4\pi/9)$$
$$e_4 = \epsilon^4 + \epsilon^5 = 2\cos(8\pi/9)$$

TABLE 10

EFFICIENCY FACTORS FOR THE DESIGN IN EXAMPLE 7.1

α	χ_0	χ_1	χ_2	χ_3	χ_4
μ	1	0	0	0	0
σ	0	0	0	0	0
ρ	0	0	0	$\frac{1}{4}$	0
κ	0	$\dfrac{3+e_2-e_1}{36}$	$\dfrac{3+e_4-e_2}{36}$	0	$\dfrac{3+e_1-e_4}{36}$
ε	0	$\dfrac{33-e_2+e_1}{36}$	$\dfrac{33-e_4+e_2}{36}$	$\frac{3}{4}$	$\dfrac{33-e_1+e_4}{36}$

With the elements of Γ in the order $\mu, \sigma, \rho, \kappa, \varepsilon$, the Möbius function of Γ is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Thus Theorem 7.3 and Table 9 give the efficiency factors of the design, which are shown in Table 10.

EXAMPLE 7.2. In dairy hygiene experiments, different cleaning treatments may be used on the two sides of the milking parlour at each farm, and treatments are typically changed monthly. This gives the simple orthogonal block structure

$$(\text{farms}/\text{sides}) \times \text{months}$$

in the notation of [61]. Label the partitions into farms, sides, months, farm-months by $\psi, \sigma, \rho, \kappa$ respectively. Suppose that there are six farms and four months, so that $|\Omega| = 48$. Let G be the Abelian group $\mathbf{Z}_2 \times \mathbf{Z}_6$, written

TABLE 11

INITIAL ψ-BLOCK FOR THE DESIGN IN EXAMPLE 7.2

00	11
10	01
14	05
04	15

TABLE 12

VALUES OF n_g FOR THE DESIGN IN EXAMPLE 7.2

Block system	n_g							
	$g=00$	10	03	13	± 01	± 02	± 11	± 12
ρ	12	12	12	12	12	12	12	12
ψ	8	8	4	4	6	4	6	4
σ	4	4	0	0	0	2	0	2
κ	2	0	0	0	0	0	1	0

additively in the abbreviated notation of [32]. Let Δ be the design generated by G from the initial ψ-block in Table 11, where rows denote months and columns denote sides. Values of n_g are in Table 12.

From [58, Theorem 2.4], the irreducible characters of G are

$$\{\chi_{h_1h_2}: h_1 \in \mathbf{Z}_2, h_2 \in \mathbf{Z}_6\},$$

where

$$\chi_{h_1h_2}(g_1, g_2) = \epsilon^{3h_1g_1+h_2g_2}, \qquad \epsilon \text{ being a primitive 6th root of 1.}$$

Table 13 shows the values of $\nu_{\gamma\chi}$.

With the block systems in the order $\mu, \rho, \psi, \sigma, \kappa, \varepsilon$, the Möbius function is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

This gives the efficiency factors in Table 14.

TABLE 13

VALUES OF $\nu_{\gamma\chi}$ FOR THE DESIGN IN EXAMPLE 7.2

γ	χ_{00}	χ_{10}	χ_{03}	χ_{13}	χ_{01}	χ_{02}	χ_{11}	χ_{12}
μ	1	0	0	0	0	0	0	0
ρ	1	0	0	0	0	0	0	0
ψ	1	0	0	0	$\frac{3}{16}$	$\frac{1}{16}$	0	0
σ	1	0	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
κ	1	0	0	1	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$
ε	1	1	1	1	1	1	1	1

TABLE 14

EFFICIENCY FACTORS FOR THE DESIGN IN EXAMPLE 7.2

α	χ_{00}	χ_{10}	χ_{03}	χ_{13}	χ_{01}	χ_{02}	χ_{11}	χ_{12}
μ	1	0	0	0	0	0	0	0
ρ	0	0	0	0	0	0	0	0
ψ	0	0	0	0	$\frac{3}{16}$	$\frac{1}{16}$	0	0
σ	0	0	1	0	$\frac{1}{16}$	$\frac{3}{16}$	0	0
κ	0	0	0	1	$\frac{9}{16}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{4}$
ε	0	1	0	0	$\frac{3}{16}$	$\frac{9}{16}$	$\frac{3}{4}$	$\frac{1}{4}$

In Example 7.2 the efficiency factors are all rational, and this is a consequence of the remark at the end of Section 6. In general, if e divides 12, then the G-homogeneous decomposition given in Theorem 6.1 is identical to that obtained from the group block structure on G defined by the complete subgroup lattice Γ' of G. The matrices $(Q_\chi)_{\chi \in G^*}$ of orthogonal projection onto the spaces $(W_\chi)_{\chi \in G^*}$ are thus obtained as in Equation (2) from blocks-averaging matrices on $G \times G$ by means of the Möbius function of Γ'; they thus have rational entries. Similarly, the information matrices $(L_\alpha)_{\alpha \in A}$ have rational entries. Now the efficiency factor $\lambda_{\alpha\chi}$ is a rational multiple of the trace of $L_\alpha Q_\chi$, and so it is rational.

Finally, we show how the results of this section specialize to give the results of [7] and [32].

DEFINITION. The treatment subspace W_χ is *totally confounded* in stratum $\mathscr{S}_\alpha$ if $\lambda_{\alpha\chi} = 1$.

DEFINITION. If H is a subgroup of the Abelian group G, then the *annihilator* H^0 of H is defined by

$$H^0 = \{\chi \in G^* : \chi(h) = 1 \text{ for all } h \text{ in } H\} \qquad (\text{see } [7]).$$

THEOREM 7.7. *Suppose that $(\Omega, \Gamma, G, \phi)$ is a homogeneous (G, γ)-design. Let b be one of the initial γ-blocks, and suppose that b has multiset K. Let H be the stabilizer in G of K, and let N be the group generated by all quotients tu^{-1} for treatments t, u such that $K(t)K(u) > 0$. Then*

(i) $N^0 \subseteq H^0$;
(ii) *if* $\chi \in N^0$ *then* $\nu_{\gamma\chi} = 1$;
(iii) *if* $\chi \notin H^0$ *then* $\nu_{\gamma\chi} = 0$.

Proof. (i): The multiset K is constant on each coset of H in G (compare this with [32, Theorem 1], where H is defined to be the largest subgroup with this property and then effectively proved to be the stabilizer of K). If Hg is any coset of H in G, then $\{tu^{-1}: t \in Hg \text{ and } u \in Hg\} = H$. Thus $H \subseteq N$. Taking annihilators reverses inclusion, so $N^0 \subseteq H^0$.

(ii): If $g \notin N$ then $n_g = 0$. Thus

$$\nu_{\gamma\chi} = \frac{1}{k_\gamma^2} \sum_{g \in N} n_g \chi(g).$$

If $\chi \in N^0$ then

$$\sum_{g \in N} n_g \chi(g) = \sum_{g \in N} n_g = \sum_{g \in G} n_g = k_\gamma^2$$

and so $\nu_{\gamma\chi} = 1$.

(iii): Since K is constant on cosets of H, the value of n_g depends only on the coset of H containing g, and so $\nu_{\gamma\chi}$ is a multiple of $\sum_{h \in H} \chi(h)$. If $\chi \notin H^0$, then the restriction of χ to H is not the principal character of H, so Equation (6) shows that $\sum_{h \in H} \chi(h) = 0$, and so $\nu_{\gamma\chi} = 0$. ∎

Since the annihilator H^0 is naturally isomorphic to the dual of the quotient group G/H [42, Section V.6], there is another way to view part (iii) of Theorem 7.7. If the block stabilizer H is a nontrivial subgroup of G, then a G-design may be constructed in two stages: first a design for $|G|/|H|$ treatments is generated using G/H; then each treatment is replaced by $|H|$ new treatments. All contrasts within cosets of H are orthogonal to γ-blocks, and the efficiency factors for the contrasts between cosets of H are the same as those for the original design on G/H.

Part (ii) of Theorem 7.7 has a similarly straightforward interpretation, because the contrasts between cosets of N are totally confounded (or possibly superconfounded, in the sense of [7]) with γ-blocks.

In some Abelian group designs (for example, [51, 31, 7]) there are subgroups $(H_\gamma)_{\gamma \in \Gamma}$ of G with the following property: if b is any γ-block, then there is some coset $H_\gamma g$ of H_γ such that the multiset of b consists of a number of copies of $H_\gamma g$. In this case $H_\gamma = N_\gamma$ for all γ in Γ, where N_γ is defined like the subgroup N in Theorem 7.7. Hence

$$\nu_{\gamma\chi} = \begin{cases} 1 & \text{if } \chi \in H_\gamma^0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, in such designs, every treatment subspace W_χ is totally confounded in some stratum.

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