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Factorial Design and Abelian Groups

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ABSTRACT

The theory of finite Abelian groups is used to simplify the search for, and construction of, factorial designs.

1. INTRODUCTION

The classical theory of symmetrical factorial design, where all treatment factors have the same, prime number of levels [9, 11], was expressed in terms of elementary Abelian groups by Fisher [20, 21] and Finney [18]. Chakravarti [13] explicitly used the dual group in the case of 2-level factors. More recently, several authors have used *general* finite Abelian groups to extend the classical theory of factorial design to include the cases where not all treatment factors have the same number of levels and where the number(s) of levels is (are) not necessarily prime: see [1, 4, 7, 16, 17, 25, 26, 30, 31].

Much of this work is concerned with fractional replicates, or with single replicates divided into equal-sized blocks. However, there is increasing interest in more complicated block structures, such as row-by-column layouts or multiply nested structures: some important classes of block structure are (ordered by inclusion) the *simple orthogonal block structures* [32], the *poset block structures* [2, 5, 8, 38, 39], the *orthogonal block structures* [5, 38, 39], and the *Tjur block structures* [41]. In spite of the statistical overtones of the word "block," both the set of plots and the set of treatments may be equipped with a "block structure" of some sort. General block structures therefore provide the natural setting for the decomposition of both the treatment space and the plot space of a factorial experiment. Use of these general block structures makes essential the explicit clarification of the associated vector-space theory. This theory has been developed by several authors, with conflicting terminology and notation. To avoid confusion in the

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main part of this paper, the necessary vector-space theory is reviewed in Section 2.

In Section 3 we set some previous design theory in the context of more general block structures. We briefly review how the identification of the sets of treatments and plots with Abelian groups can, for designs defined by group homomorphisms, facilitate the description and examination of confounding patterns. Section 4 contains the main results of this paper. It shows how some group theory, specifically the decomposition of a finite Abelian group into a direct sum of its Sylow subgroups, may be used to simplify many design problems. Thus it suffices to consider Abelian groups of prime-power order. Section 5 specializes these results to fractional replicates and blocked single replicates. For these two cases, it is shown that design problems for general finite Abelian groups may be essentially reduced to similar problems for elementary Abelian groups.

Unless otherwise explained, all theory, terminology, and notation for groups used in this paper may be found in [23]. A similar reference work for factorial design is [27].

2. BLOCK STRUCTURES AND VECTOR-SPACE DECOMPOSITIONS

Let Ω be a finite set, let ρ and σ be partitions of Ω (that is, equivalence relations on Ω, or block systems on Ω), and let Σ be a set of partitions of Ω. We use the following terminology and notation for concepts given in [2, 4, 5, 8, 28, 38, 39, 40, 41]:

(1) ρ is *uniform* if all classes of ρ have the same size;

(2) V_ρ denotes the subspace of the real vector space $\mathbb{R}^\Omega$ consisting of vectors which are constant on each ρ-class;

(3) ρ *nests* σ, or ρ is *coarser* than σ, or σ is *finer* than ρ, if every σ-class is contained in a ρ-class (equivalently, if $V_\rho \leqslant V_\sigma$);

(4) $\rho \wedge \sigma$ and $\rho \vee \sigma$ are, respectively, the coarsest partition nested by both ρ and σ, and the finest partition which nests both ρ and σ;

(5) ρ and σ are *orthogonal* if $V_\rho \cap V_{\rho\vee\sigma}^\perp$ is orthogonal to $V_\sigma \cap V_{\rho\vee\sigma}^\perp$ [equivalently, if, within each $(\rho \vee \sigma)$-class, the size of each $(\rho \wedge \sigma)$-class is proportional to the product of the sizes of the ρ-class and σ-class containing it];

(6) μ and ε are, respectively, the coarsest and finest partitions of Ω;

(7) if $\sigma \in \Sigma$, then W_σ is the orthogonal complement in V_σ of all those V_ρ for which $\rho \in \Sigma$, $\rho \neq \sigma$, and ρ nests σ;

(8) (Ω, Σ) is a *Tjur block structure* (TBS) if Σ contains ε, Σ is closed under $\vee$, and every pair of partitions in Σ is orthogonal;

(9) (Ω, Σ) is an *orthogonal block structure* (OBS) if it is a TBS which is closed under $\wedge$, which contains μ, and all of whose partitions are uniform.

THEOREM 1 (see [41]). *If (Ω, Σ) is a Tjur block structure, then $\mathbb{R}^{\Omega}$ is the orthogonal direct sum of the spaces W_σ for σ in Σ.*

Three special classes of OBS are of particular interest to us here. The first has been implicitly described by almost every author on factorial design. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$ for some integer n, and let Σ be the set of subsets of $\{1,2,\ldots,n\}$. If $\sigma \in \Sigma$, we may identify σ with a partition of Ω by putting ω and ω' in the same σ-class if and only if $\omega_i = \omega_i'$ for all i in σ. In particular, $\varnothing = \mu$ and $\{1,2,\ldots,n\} = \varepsilon$. Then (Ω, Σ) is an OBS, and the decomposition $\oplus_\sigma W_\sigma$ is the usual factorial decomposition of $\mathbb{R}^{\Omega}$, for $W_{\{i\}}$ is the (space corresponding to the) main effect of the ith factor, while W_σ is the interaction of the factors in σ for $|\sigma| \geqslant 2$. We call this a *complete factorial structure.*

The second special class is a generalization of the first. Suppose that $\preccurlyeq$ is a partial order on $\{1,\ldots,n\}$. A subset σ of $\{1,\ldots,n\}$ is said to be *ancestral* if whenever $i \in \sigma$ and $i \preccurlyeq j$ then $j \in \sigma$. Let $\Sigma_{\preccurlyeq}$ be the set of all ancestral subsets of $\{1,\ldots,n\}$. Then $(\Omega, \Sigma_{\preccurlyeq})$ is an OBS. This is called a *poset block structure* (PBS).

The third special class consists of the *group block structures* (GBS). Let G be an Abelian group. If K is a subgroup of G, we define the partition σ_K of G to be that whose classes are the cosets of K. Then $\sigma_K \wedge \sigma_L = \sigma_{K \cap L}$, $\sigma_K \vee \sigma_L = \sigma_{K+L}$, $\sigma_G = \mu$, and $\sigma_0 = \varepsilon$: here we are writing G additively, so that 0 denotes both the identity element of G and the identity subgroup. If $\Sigma = \{\sigma_K : K \leqslant G\}$, then (G, Σ) is an OBS.

It is convenient to describe GBS in another way. Every finite Abelian group G has a *dual* group G^* (see [23, §V.6]), consisting of the homomorphisms from G into the multiplicative group of the complex numbers. Moreover, G may be expressed (in general, in more than one way) as a direct sum of cyclic groups $\langle g_1 \rangle \oplus \langle g_2 \rangle \oplus \cdots \oplus \langle g_m \rangle$, and this gives an explicit form for G^*, as Ledermann [29, §2.4] shows. Let t_i be the order of g_i, and let c be any common multiple of the t_i. Then $G^* = \langle \chi_1 \rangle \oplus \langle \chi_2 \rangle \oplus \cdots \oplus \langle \chi_m \rangle$, where

$$\chi_i(g_j) = \eta^{\delta_{ij} c / t_i}$$

and η is a primitive cth root of unity. Thus if z in G and θ in G^* are equal to $\Sigma_i z_i g_i$ and $\Sigma_i \theta_i \chi_i$ respectively, where the z_i and θ_i are integers, then

$$\theta(z) = \eta^{[\theta, z]}$$

where [,] is the bilinear form given in [7], namely

$$[\theta, z] = \sum_i \frac{\theta_i z_i c}{t_i}.$$

For a subgroup K of G, we denote by K° the *annihilator* of K; that is, $K^\circ = \{\chi \in G^* : \chi(k) = 1 \text{ for all } k \in K\}$. By Theorems 6.3 and 6.4 of [23, §V.6], we have $G^{**} = G$ and $K^{\circ\circ} = K$. If H is a subgroup of G^*, we define the partition ρ_H of G as follows:

z and z' are in the same ρ_H-class if and only if

$$\chi(z) = \chi(z') \text{ for all } \chi \in H.$$

Then $\rho_H = \sigma_{H^\circ}$ so that $(G, \{\rho_H : H \leqslant G^*\})$ is the same GBS that we described before, except that the partitions are now labeled by subgroups of G^* rather than subgroups of G. It is more convenient for subsequent work to use this second labeling, and we shall write H for ρ_H where no confusion can arise: in particular, we write W_H for W_{ρ_H}.

The group G is sometimes given in the form $\Omega_1 \oplus \cdots \oplus \Omega_n$, where the Ω_i are groups and do not necessarily bear any relationship to the $\langle g_j \rangle$. If $\chi \in G^*$, then χ has a unique expression as $\sum_i \xi_i$, where $\xi_i \in \Omega_i^*$. Define the subset $J(\chi)$ of $\{1, \ldots, n\}$ by $J(\chi) = \{i : \xi_i \neq 0\}$. Thus we have two decompositions of $\mathbb{R}^G$, one given by the group block structure on G and one given by the complete factorial structure on G. The results of [7] link these two decompositions, and may be stated algebraically as follows.

THEOREM 2. *Let $H \leqslant G^*$. Then*

(a) $W_H = 0$ *unless H is cyclic;*
(b) *if H is cyclic, with generator χ, then*
  (b1) *the dimension of W_H is equal to $\varphi(|H|)$, where φ is Euler's function;*
  (b2) $W_H \leqslant W_{J(\chi)}$.

3. FACTORIAL DESIGN AND CONFOUNDING

Let P be a set of plots with a TBS (P, Γ). The subspaces W_γ of $\mathbb{R}^P$, for γ in Γ, are called *strata* [32].

For reasons given in the discussion in [41], (P, Γ) will usually be an OBS. If a randomization argument is used to justify the assumed linear model (see [3, 32]), then (P, Γ) will almost always be a PBS (see [2, 8]). If (P, Γ) is a PBS with $P = P_1 \times \cdots \times P_n$, then P may be given the structure of an Abelian group in the following canonical way. Identify each P_i with any Abelian group of order $|P_i|$, so that $P = P_1 \oplus \cdots \oplus P_n$. Each γ in Γ is an ancestral subset of $\{1, \ldots, n\}$: for each such γ define the subgroup K_γ of P by $K_\gamma = \oplus \{P_i : i \notin \gamma\}$. If we define σ_{K_γ} as for the group block structure on P, then σ_{K_γ} is the partition corresponding to γ.

From now on, we restrict our attention to TBS (P, Γ) with the property that P may be given the structure of an Abelian group in such a way that every γ in Γ is equal to some σ_K for $K \leqslant P$. As just shown, this is always possible if (P, Γ) is a poset block structure: see Example 1. Even for a PBS, the subgroups K may not be the canonical ones just given: see Example 2. Moreover, we do not exclude other TBS with this property, such as some of the Latin-square block structures in [36, §5] (see Example 3), or any subset, closed under $\vee$, of the partitions in a PBS (see Example 4).

Example 1. Let (P, Γ) be the simple orthogonal block structure ((6 rows $\times$ 6 columns)/6 subplots) (in the notation of [32]). We may put $P = \langle p_r \rangle \oplus \langle p_c \rangle \oplus \langle p_s \rangle$, where p_r, p_c, and p_s have order 6. The subgroups corresponding to μ, rows, columns, rows $\wedge$ columns, and ε are P, $\langle p_c \rangle \oplus \langle p_s \rangle$, $\langle p_r \rangle \oplus \langle p_s \rangle$, $\langle p_s \rangle$, and 0 respectively.

Example 2. Let (P, Γ) be the simple orthogonal block structure ((2 superblocks/2 blocks/4 plots)). We may put $P = \langle p_1 \rangle \oplus \langle p_2 \rangle$, where p_1 and p_2 have order 4, and let $P, \langle 2p_1 \rangle \oplus \langle p_2 \rangle, \langle p_2 \rangle, 0$ be the subgroups corresponding to μ, superblocks, blocks, ε respectively.

Example 3. Let $|P| = 81$, and let the nontrivial partitions on P be the rows, columns, and letters of a Latin square Λ based on the elementary Abelian group G of order 9 (for example, Λ may be any 9×9 square in Table XVI of [22]). This structure is an OBS but not a PBS. By suitably labeling rows and columns we may put $P = G \oplus G$ and take $0 \oplus G$, $G \oplus 0$ and $\{(g, -g) : g \in G\}$ to be the subgroups corresponding to rows, columns, and letters respectively.

Example 4. As in [37, §4(d)], let P be an $a \times b \times c$ 3-dimensional array whose only nontrivial partitions are the 2-dimensional slices corresponding to main effects of the three coordinates. This is a TBS but not an OBS. Put $P = P_1 \oplus P_2 \oplus P_3$, where P_1, P_2, P_3 are Abelian groups of order a, b, c respectively. The nontrivial partitions correspond to the subgroups $P_2 \oplus P_3$, $P_1 \oplus P_3$, and $P_1 \oplus P_2$.

Let T be the set of treatments to be applied to P. In a factorial experiment with n treatment factors, T is naturally $T_1 \times \cdots \times T_n$, where T_i is the set of levels of the ith factor. If each T_i is given an Abelian group structure, then T becomes the Abelian group $T_1 \oplus \cdots \oplus T_n$. If $t_i = |T_i|$, then the most natural choice for T_i is the cyclic group C_{t_i} of order t_i, and this is the approach taken in [16, 26, 30, 31]. However, there are other possibilities when t_i is not square-free, which are discussed at the end of Section 4.

Any allocation of treatments to plots is simply a function $\phi: P \to T$. A design with plot structure (P, Γ), treatment structure (T, Δ), and allocation ϕ is defined to be *isomorphic* to a design $((P', \Gamma'); (T', \Delta'); \phi'))$ if there exist bijections $\theta: P \to P'$, and $\psi: T \to T'$ such that:

(1) $\Gamma' = \{\theta(\gamma): \gamma \in \Gamma\}$, where γ is regarded as a subset of $P \times P$ and $\theta(\gamma) = \{(\theta(p_1), \theta(p_2)): (p_1, p_2) \in \gamma\}$;

(2) $\Delta' = \{\psi(\delta): \delta \in \Delta\}$;

(3) $\phi' = \psi \circ \phi \circ \theta^{-1}$.

Thus isomorphic designs differ only in the labeling of the plots, the treatments, and the partitions thereof.

We restrict attention to designs in which ϕ is a group homomorphism. This includes all *design keys* as originally defined [7, 33, 35]; for these designs (P, Γ) is a PBS with canonically chosen subgroups, and ϕ may be specified as a matrix with respect to irredundant generators of T and P. However, more general design keys [34] are not included. Moreover, some designs obtained from group homomorphisms are not isomorphic to any designs constructed by the design-key method.

EXAMPLE 5. Let $T = T_1 \oplus T_2$, where $T_i = \langle t_i \rangle$ and t_i has order 4 for $i = 1, 2$. Let (P, Γ) be as in Example 2. The group homomorphism $\phi: P \to T$ defined by $\phi(ap_1 + bp_2) = (a + b)t_1 + bt_2$ gives the design in Table 1, where

TABLE 1

a	$b = 0$	1	2	3		
0	(0,0)	(1,1)	(2,2)	(3,3)	Block 1	Superblock I
2	(2,0)	(3,1)	(0,2)	(1,3)	Block 2	
1	(1,0)	(2,1)	(3,2)	(0,3)	Block 3	Superblock II
3	(3,0)	(0,1)	(1,2)	(2,3)	Block 4	

the treatment $ct_1 + dt_2$ is shown as (c, d). This design cannot be obtained by using a design key.

When ϕ is a group homomorphism the general theory of dual structures [14, §8.2; 24, §VII.8] shows that there is a dual homomorphism $\phi^* : T^* \to P^*$ defined by

$$(\phi^*(\chi))(p) = \chi(\phi(p)) \qquad \text{for } p \text{ in } P \text{ and } \chi \text{ in } T^*,$$

and a dual-type linear transformation $\phi_* : \mathbb{R}^T \to \mathbb{R}^P$ defined by

$$(\phi_*(v))_p = v_{\phi(p)} \qquad \text{for } p \text{ in } P \text{ and } v \text{ in } \mathbb{R}^T.$$

For any subgroup H of T^*, and any γ in Γ, the *treatment effect* W_H is said to be *confounded* with γ if $\phi_*(W_H) \leqslant W_\gamma$, and to be *superconfounded* [1] with γ if $\phi_*(W_H) \leqslant V_\gamma$. Define the subset L_γ of T^* by $L_\gamma = \{\chi \in T^* : W_{\langle \chi \rangle}$ is superconfounded with $\gamma\}$. Although previously stated only for simple orthogonal block structures (P, Γ), the results of [1, 7] apply to the more general structures considered here and may be summarized as follows.

THEOREM 3. *If ϕ is a group homomorphism and H is a subgroup of T^*, then*

(a) $\phi_*(V_H) = V_{\phi^*(H)}$;
(b) $\phi_*(W_H) = W_{\phi^*(H)}$;
(c) *if $\gamma \in \Gamma$ and K_γ is the subgroup of P such that $\gamma = \sigma_{K_\gamma}$, then L_γ is the subgroup $\phi^{*-1}(K_\gamma^\circ)$ of T^*, and the treatments occurring on any γ-class of P consist of one or more copies of a coset of L_γ° in T.*

Parts (a) and (b) of Theorem 3 show the power and simplicity of the group homomorphism method of constructing designs and justify the stratum identification rule stated in [35]. However, not all designs based on Abelian groups can be constructed by a design key: those in [17] cannot, in general. The group-theoretical ideas of the present paper may be extended to these more general designs by using the finer decomposition of the W_H mentioned at the end of [4], and this is done in a subsequent paper.

Theorem 3(c) shows that the group homomorphism itself may be bypassed, and the design constructed by choosing suitable, and compatible, subgroups L_γ and calculating their annihilators. This is the approach given in [1] and the discussion of [41], and the one that will be followed in much of the remainder of this paper.

4. PSEUDOFACTORS AND SYLOW SUBGROUPS

Many authors (for example, Yates [43] and Patterson [33]) have advocated the use of pseudofactors in constructing factorial designs where any t_i is not prime. Suppose that $t_i = r_i s_i$, where r_i and s_i are positive integers greater than 1. Then the ith factor may be replaced by a pair of pseudofactors with r_i and s_i levels. The set T_i is identified with $R_i \times S_i$, where $|R_i| = r_i$ and $|S_i| = s_i$: any convenient one-one correspondence between the levels of the ith factor and the pairs of levels in $R_i \times S_i$ may be chosen.

The simplest way of reconciling the straightforward convention of [16, 26] (that each T_i is identified with a cyclic group) with the flexibility of allowing each T_i to be an arbitrary Abelian group of order t_i, is to permit use of pseudofactors, with the convention that each pseudofactor or unchanged factor is represented by a cyclic group. Does such use of pseudofactors make any difference to the designs which may be constructed?

If there are no pseudofactors, T and T^* are both isomorphic to $C_{t_1} \oplus \cdots \oplus C_{t_n}$. Replacement of the ith pseudofactor by two pseudofactors with r_i and s_i levels (where $r_i s_i = t_i$) has the effect of replacing the direct summand C_{t_i} by $C_{r_i} \oplus C_{s_i}$. If r_i and s_i are coprime, these two groups are isomorphic [23, Theorem I.13.9; and 14, §9.6] and so there is no change in the group structure, and hence no change in the designs produced by the group homomorphism or annihilator methods.

Repeated application of this argument gives the following result.

THEOREM 4. *If each treatment factor is replaced by pseudofactors, each of which has a prime-power number of levels, there being one pseudofactor for each prime which divides the original number of levels, then there is no change in the designs constructed by the group homomorphism or annihilator methods.*

Let Q be the set of primes dividing $|T|$. By Theorems 7.3, 7.5, 9.5, and 13.9 of [23, §I], or Proposition 9.7.4 of [14], T^* has a unique Sylow q-subgroup T_q^* for each q in Q, and T^* is the direct sum $\oplus_{q \in Q} T_q^*$. Moreover, if H is any subgroup of T^*, then $H = \oplus_{q \in Q} H_q$, where $H_q \leqslant T_q^*$. Similar results hold for T, P, P^*, and their subgroups [since duals of direct sums are the direct sums of the duals, there is no ambiguity about the notation T_q^* for $(T_q)^* = (T^*)_q$]. For each prime q in Q the Sylow q-subgroups of T and T^* correspond to the *treatment q-quotient structure* consisting of those factors and pseudofactors whose numbers of levels are powers of q (see [15, §4] for why the term "quotient structure" is more appropriate than the term "substructure" used by some authors).

If $\gamma \in \Gamma$, then γ corresponds to some subgroup of P. Hence we obtain a partition γ_q of the plot q-quotient structure. If γ and δ are distinct elements of Γ, then γ_q and δ_q may be equal for some (but not all) values of q: in particular, γ_q may be equal to μ_q or to ε_q for some values of q. Thus the block structure (P_q, Γ_q) may be less rich than (P, Γ): technically, it is a *quotient* block structure of (P, Γ).

Suppose that, for each q, we have a design for treatment set T_q on plot set P_q. In the case that each design is a fractional or single replicate, with no blocking, Chakravarti [12] suggested *combining* the separate designs by taking the direct product of the sets of treatments which occur in the separate designs. We may generalize this idea to more complicated designs. For q in Q, suppose that $\phi_q : P_q \to T_q$ is a treatment allocation (not necessarily a group homomorphism). Since, as sets, $P = \prod_{q \in Q} P_q$ and $T = \prod_{q \in Q} T_q$, there is a natural way of defining the function ϕ as the direct product of the ϕ_q:

$$(\phi(p))_q = \phi_q(p_q) \qquad \text{for} \quad p \in P.$$

Equivalently, the γ-classes of P are all the possible direct products $\prod_{q \in Q} B_q$, where B_q is a γ_q-class of P_q, and the treatments on $\prod B_q$ are all combinations (allowing for multiplicities) of the treatments on the constituents B_q.

An immediate consequence of Theorem V.6.4 of [23] is that the annihilator of a direct sum is the direct sum of the annihilators: thus if $H \leqslant T^*$ then $H^\circ = \oplus_q H_q^\circ$. This gives our main result.

THEOREM 5. *Constructing a design for treatment set T on block structure (P, Γ) by the group homomorphism or annihilator methods is equivalent to constructing such a design for T_q on (P_q, Γ_q) for each q in Q and then combining these designs.*

This shows that, in any search for suitable designs with the complete confounding guaranteed by the group homomorphism method, we may essentially restrict our attention to *q-designs*, those in which $|P|$ and $|T|$ are both powers of the same prime q. Thus any *given* treatment structure may be decomposed into its constituent q-quotient structures (which are usually easier to deal with), for any investigation of which fractions, block designs, etc. are possible by the annihilator method. This decomposition was advocated in [35, §7], and could have been used to facilitate the searches for designs in [16, 31].

For fractional designs, we use the definition of *resolution* given in [27, §8.2], so that a resolution-w fraction also has resolution $w-1$. Define the *weight* of χ in T^* to be $|J(\chi)|$, where $J(\chi)$ is defined with respect to the decomposition $T_1 \oplus \cdots \oplus T_n$, as in Section 2; and define a subgroup H of T^* to be *w-heavy* if all its nonzero elements have weight at least w. Since the defining contrasts of a fractional replicate are precisely the elements of L_μ, a fraction has resolution w if and only if L_μ is w-heavy. Chakravarti [12] pointed out that a fraction has resolution w if and only if each of its constituent q-designs has resolution w.

The concept of resolution may be usefully generalized to other block structures, because requirements such as "no main effect or two-factor interaction should be confounded with blocks" are quite common. If w is any function from Γ to the positive integers $\mathbb{Z}^+$, we say that a design has resolution w if, for all γ in Γ, the group L_γ is w_γ-heavy. An easy generalization of Chakravarti's result is:

THEOREM 6. *Let w be a function from Γ to $\mathbb{Z}^+$. A design on (P, Γ) has resolution w if and only if each constituent q-design has resolution w.*

The notation used so far is convenient for general theory, but cumbersome for examples. Thus for the remaining examples we denote the treatment factors by A, B, C,... as usual: there is no harm in identifying these with $\chi_1, \chi_2, \chi_3, \ldots$. Pseudofactors are shown by appending subscripts to the corresponding genuine factors. Since T and T^* are written additively, it is logical to write generalized factors in the form exemplified by $A + 2B + C$, rather than the classical AB^2C: although this notation is different from that in most of the literature, and takes more space, it has the advantages that it is more consistent with the remaining notation, and that generalized factors such as $A + B + C$ are clearly distinguished from interactions such as W_{ABC}; my students have certainly found the additive notation easier to follow. As in [6], the space $W_{\langle \chi \rangle}$ is written as $[\chi]$.

EXAMPLE 6. Consider a single-replicate 6^3 design in the block structure ((6 rows $\times$ 6 columns)/6 subplots) discussed in Example 1. Let A_2, B_2, C_2 and A_3, B_3, C_3 be the pseudofactors with 2 and 3 levels respectively. The constituent 2-design is for 2^3 treatments in $(2 \times 2)/2$. In this, the one-dimensional space $[A_2 + B_2 + C_2]$ is the whole of the 3-factor interaction. For a single replicate, the two one-dimensional spaces $\phi_*^{-1}(W_{\text{rows}})$ and $\phi_*^{-1}(W_{\text{columns}})$ must be different. Hence there can be no single replicate 6^3 design in $(6 \times 6)/6$ in which the effects confounded with rows or with columns belong entirely to the 3-factor interaction.

A design with minimal confounding of 2-factor interactions with rows or columns has $[A_2 + B_2 + C_2]$ and $[A_2 + C_2]$ confounded with rows and columns respectively in the 2-design, and $[A_3 + B_3 + C_3]$ and $[A_3 + B_3 + 2C_3]$ confounded with rows and columns respectively in the 3-design. However,

$[B_2]$ and $[C_3]$, which are parts of main effects, are then confounded with rows $\wedge$ columns. In order to have all main effects estimated in the subplots stratum, one would have to use a design with a confounding pattern such as

	rows	columns	rows $\wedge$ columns
2-design	$[A_2+B_2]$	$[B_2+C_2]$	$[A_2+C_2]$
3-design	$[A_3+B_3+C_3]$	$[A_3+2B_3]$	$[A_3+2C_3], [B_3+2C_3]$

Table 1 of [16] is a list of resolution-II single-replicate block designs for asymmetrical factorial structures, with $|T| \leqslant 56$, $\max(t_i) \leqslant 7$, and $n \leqslant 5$. Table 1 of [31] lists resolution-III fractions (or single-replicate block designs) for asymmetrical factorial structures with $|T| \leqslant 200$, $\max(t_i) \leqslant 7$, and $n \leqslant 7$. In all but three cases, every design with a factor (or factors) at 6 levels may be obtained from another design in the same table by using pseudofactors. The three exceptions are interesting. The $\frac{1}{4}$-replicate $2^3 \times 4 \times 6$ and $2^4 \times 3 \times 4$ designs in [31] are not isomorphic, because their 2-quotient structures are not isomorphic. As we show in Section 5, a resolution-III $\frac{1}{4}$-replicate $2^4 \times 4$ design may essentially have defining contrasts $A+B+C$, $A+D+2E$, $B+C+D+2E$ or $A+B+C+D$, $A+B+2E, C+D+2E$.

[16] contains designs for (a) $2^3 \times 6$, (b) $2^4 \times 3$, (c) $2^2 \times 3 \times 4$, (d) $2 \times 4 \times 6$ in 2 blocks of 24 plots. Design (a), with defining contrast $B+C+3D$, is not obtainable from design (b), whose defining contrast is $A+B+C+D$: use of design (b) and pseudofactors gives a $2^3 \times 6$ design with defining contrast $A+B+C+3D$, which is of higher resolution than design (a), contrary to the authors' claims. Similarly, designs (c) and (d) have defining contrasts $B+2D$ and $A+2B+3C$ respectively: the $2^2 \times 3 \times 4$ design corresponding to design (d) has defining contrast $A+B+2D$, and so is better than design (c). The design given for $2^3 \times 4$ in 2 blocks of 16 plots is also not of highest possible resolution. However, these mistakes are all corrected in [31].

The question posed at the beginning of this section has not yet been answered for the case when r_i and s_i are not coprime. Then C_{t_i} is *not* isomorphic to $C_{r_i} \oplus C_{s_i}$, so the use of pseudofactors may lead to designs which are not isomorphic to those obtained without pseudofactors, except when $T_i \cap L^\circ_\gamma = 0$ or T_i for every γ. Since an elementary Abelian group has more subgroups, of all orders, than any other group of the same order, replacement of a factor with q^r levels (where q is prime) by r pseudofactors gives more flexibility in choice of design, and may make possible designs which cannot be obtained without pseudofactors.

EXAMPLE 7. Suppose that we require a resolution-III $\frac{1}{4}$-replicate of a 4^3 treatment structure. Without pseudofactors, there are only 4 possible defining contrasts subgroups L_μ; one is $\langle A+B+C\rangle$, confounding $[A+B+C]$ and $[2A+2B+2C]$ with μ. The corresponding designs are isomorphic, each comprising the 16 (row, column, letter)-triplets of the cyclic 4×4 Latin square. With pseudofactors, there are 36 distinct, but isomorphic, fractions; one of the possible defining contrasts subgroups is $\{0, A_1+B_1+C_1+C_2, A_2+B_2+C_1, A_1+A_2+B_1+B_2+C_2\}$. These fractions correspond to the other species of 4×4 Latin squares, and so are not isomorphic to the previous fractions.

EXAMPLE 8. The list in [31] shows no $\frac{1}{4}$-replicate $2^2\times 4^2$ design, because no such resolution-III fraction is available without pseudofactors. Using pseudofactors, there is a resolution-III $\frac{1}{4}$-replicate with defining contrasts $A+C_1+D_1$, $B+C_2+D_2$, $A+B+C_1+C_2+D_1+D_2$.

EXAMPLE 9. For a 4^2 treatment structure, suppose that a resolvable design [27, §11.4] is required in 3 replicates of 4 blocks of 4 plots. Using pseudofactors, and a separate group homomorphism in each replicate, we may take the group L_{blocks} to be as follows in the 3 replicates:

(1) $\{0, A_1+B_1, A_2+B_2, A_1+A_2+B_1+B_2\}$,
(2) $\{0, A_1+B_2, A_2+B_1+B_2, A_1+A_2+B_1\}$,
(3) $\{0, A_1+B_1+B_2, A_2+B_1, A_1+A_2+B_2\}$,

thus obtaining *factorial balance* [42]. (This design is isomorphic to one obtainable by the methods of Bose [9] using the finite field of order 4.) Without pseudofactors, $2A+2B$ would have to be confounded in every replicate, making its estimation in the plots stratum impossible.

The only disadvantage of using pseudofactors with the same number of levels is one not of practicality but of enumeration: there are no longer straightforward rules, such as those given in Section 5, to determine which structures have fractions of given size and resolution. Likewise, they do not readily lend themselves to the sort of computer search undertaken in [31]. However, if, in any particular practical case, the published tables give no suitable design, it is always worth a little ad hoc investigation to see if pseudofactors can give better designs, as in Examples 8 and 9.

5. HEAVY SUBGROUPS AND DESIGNS OF SPECIFIED RESOLUTION

In this section we characterize all factorial structures T with 2-, 3-, and 4-heavy subgroups, and make some progress for w-heavy subgroups for $w \geqslant 5$. By Theorems 5 and 6, it suffices to do this for q-designs. The concept of weight makes sense only because, of the many possible ways of expressing T as a direct sum of cyclic groups, one such expression is distinguished, corresponding to the treatment factors. We shall call such a group a *named group*. Correspondingly, if T and T' are two named groups, with distinguished direct-sum decompositions $T_1 \oplus \cdots \oplus T_n$ and $T_1' \oplus \cdots \oplus T_{n'}'$ respectively, we shall call a group homomorphism ψ from T to T' a *name homomorphism* if, for $i = 1, \ldots, n$, $\psi(T_i)$ is contained in some T_j' and, for $i \neq k$, $\psi(T_i) \cap \psi(T_k) = 0$. If ψ is also a surjection, we call ψ a *name epimorphism*, T' a *name-homomorphic image* of T, and T a *name extension* of T'; if ψ is an injection, we call T a *name subgroup* of T'. (This terminology is consistent with general algebraic usage: see [15].) In factorial terms, if T has factors with $t_1, \ldots, t_n$ levels and T' has factors with $t_1', \ldots, t_n'$ levels respectively, this means that T is a name extension of T' if $n \geqslant n'$ and there is a permutation π of $\{1, \ldots, n\}$ such that t_i' divides $t_{\pi(i)}$ for $i = 1, \ldots, n'$. For example, the factorial structures $4 \times 4 \times 8 \times 8$, $2 \times 4 \times 4 \times 8$, and $4 \times 8 \times 16$ are all name extensions of $4 \times 4 \times 8$, but $2 \times 4 \times 4$ and 8×16 are not. To save cumbersome notation, we write $(q^j)^N$ for a structure with N factors at q^j levels.

THEOREM 7. *Let q be a prime number. Then there is a function $f_q : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ such that T^* has a w-heavy subgroup isomorphic to $(C_q)^e = C_q \oplus C_q \oplus \cdots \oplus C_q$ (e factors) if and only if T is a name extension of $(q)^{f_q(w,e)}$.*

Proof. If $(C_q)^N$ has a w-heavy subgroup isomorphic to $(C_q)^e$, then so does $(C_q)^{N+1}$. Moreover, the w-heavy subgroup of $(C_q)^{we}$ generated by $\xi_1, \ldots, \xi_e$, where $\xi_j = \chi_{w(j-1)+1} + \chi_{w(j-1)+2} + \cdots + \chi_{wj}$ and χ_k generates the kth C_q, is isomorphic to $(C_q)^e$. Thus there is a least integer $f_q(w,e)$ such that $(C_q)^N$ has a w-heavy subgroup isomorphic to $(C_q)^e$ if and only if $N \geqslant f_q(w,e)$.

Now let $T = T_1 \oplus \cdots \oplus T_n$, where T_i is cyclic of order q^{e_i}. Then T is a name extension of $(q)^N$ if and only if $n \geqslant N$. All elementary Abelian subgroups of T^* are subgroups of $\Omega_1(T^*)$, which is equal to $\oplus_i \langle q^{e_i - 1}\chi_i \rangle$; hence T^* has a w-heavy subgroup isomorphic to $(C_q)^e$ if and only if $n \geqslant f_q(w,e)$. ■

To find explicit forms for f_q, it suffices to consider the case when T is elementary Abelian; that is, all factors have q levels. The existence of a w-heavy subgroup H of T^* of order q^e is equivalent to the existence of a resolution-w fraction for n factors on q^{n-e} plots. Bose [9] defined $m_{w-1}(r, q)$ to be the maximum number of factors with q levels that can be accommodated in a resolution-w fraction on q^r plots. It is immediate that $f_q(w, e)$ is the smallest value of N such that $N \leqslant m_{w-1}(N-e, q)$.

THEOREM 8.

(i) $f_q(2, e) = e + 1$;
(ii) *$f_q(3, e)$ is the least integer N satisfying $N \leqslant (q^{N-e} - 1)/(q-1)$;*
(iii) *$f_2(4, e)$ is the least integer N satisfying $N \leqslant 2^{N-e-1}$;*
(iv) *$f_q(w, 1) = w$ and $f_q(w, 2) = w + \lceil w/q \rceil$, where $\lceil w/q \rceil$ is the least integer not less than w/q;*
(v) $f_q(w, e+1) \leqslant f_q(w, e) + \lceil w/2 \rceil$.

Proof. (i): Since T^* is not itself 2-heavy, $f_q(2, e) \geqslant e+1$. However, let $g = \Sigma_i g_i$, where g_i generates T_i, and put $H = \langle g \rangle^\circ$. Then $|H| = |T|/|\langle g \rangle| = q^n/q = q^{n-1}$, so $H \cong (C_q)^{n-1}$; and H is 2-heavy, because $[\theta_i \chi_i, g] \neq 0$ unless $\theta_i \chi_i = 0$. Hence $f_q(2, e) = e+1$.

(ii): By Fisher's result ([20, 21], quoted in this form by Finney [19, §4.10]), $m_2(r, q) = (q^r - 1)/(q-1)$.

(iii): Bose [9, §5] showed that $m_3(r, 2) = 2^{r-1}$.

(iv): Put $y = f_q(w, 2)$. Let H be any w-heavy subgroup of $(C_q)^y$ of order q^2. If H is also $(w+1)$-heavy, we could delete one of the named generators, contrary to the minimality of y. Hence H contains an element χ of weight w: we may suppose that $J(\chi) = \{1, \ldots, w\}$ and $\chi = \chi_1 + \chi_2 + \cdots + \chi_w$. Now let ξ be any element of $H \setminus \langle \chi \rangle$ and put $\xi = \Sigma \xi_i \chi_i$. For $j \in \{0, 1, \ldots, q-1\}$, let $\lambda_j = |\{i : 1 \leqslant i \leqslant w \text{ and } \xi_i = j\}|$. The minimality of y shows that $\xi_i \neq 0$ for $w+1 \leqslant i \leqslant y$. Hence the elements $k(j\chi - \xi)$ of $H \setminus \langle \chi \rangle$ have weights $y - \lambda_j$ for $j \in \{0, 1, \ldots, q-1\}$ and $k \in \{1, 2, \ldots, q-1\}$. Thus $y - \lambda_j \geqslant w$ for all j, so that $y - w \geqslant \max\{\lambda_j : 0 \leqslant j \leqslant q-1\}$, with equality for those families $\{\lambda_j : 0 \leqslant j \leqslant q-1\}$ which minimize this maximum. Since $\Sigma_j \lambda_j = w$, the minimum value of the maximum is $\lceil w/q \rceil$.

(v): Put $y = f_q(w, e)$ and $x = y + \lceil w/2 \rceil$. Let H be any w-heavy subgroup of $(C_q)^x$ of order q^e which involves only the first y generators. Put $\xi = \chi_{x-w+1} + \chi_{x-w+2} + \cdots + \chi_x$, where χ_k generates the kth C_q. Then $\langle H, \xi \rangle$ is a w-heavy subgroup of $(C_q)^x$ of order q^{e+1}. ∎

Explicit forms for f seem much harder to obtain for $w \geqslant 4$ and $e \geqslant 3$ (see [10]), although values of $f_q(w, e)$ for small e may be calculated by exhaustive methods.

Every Abelian q-group is a direct sum of cyclic q-groups [23, Theorem I.13.10]. Theorem 7 not only gives a necessary and sufficient condition for an Abelian q-group to contain a w-heavy subgroup isomorphic to a given *elementary* Abelian q-group; it also gives us enough information to deduce a necessary condition for an Abelian q-group to contain a w-heavy subgroup isomorphic to *any* given Abelian q-group. This is contained in Theorem 9(a). The sufficiency of this condition in some circumstances [Theorem 9(b)] is obtained by using the following lemma.

LEMMA. *Let $e_1, e_2, \ldots, e_k$ be positive integers with $e_1 < e_2 < \cdots < e_k$. Put $y_j = f_q(w, e_j)$ for $1 \leqslant j \leqslant k$. Let $T \cong (C_q)^{y_k}$, with $T^* = \oplus_{i=1}^{y_k} \langle \chi_i \rangle$. For $H \leqslant T^*$, define H_j to be $\{\chi \in H: J(\chi) \subseteq \{1, \ldots, y_j\}\}$. If* (i) *$w = 2$ or* (ii) *$w = 3$ or* (iii) *$w = 4$ and $q = 2$ or* (iv) *$e_k = 2$, then there exists a w-heavy subgroup H of T^* such that H_j is w-heavy of order q^{e_j} for $1 \leqslant j \leqslant k$.*

Proof. It suffices to assume that $e_j = j$ for $1 \leqslant j \leqslant k$. Then the proof of Theorem 8(iv) deals with the case $e_k = 2$. For the other three cases we give an explicit countable sequence $\xi_1, \xi_2, \ldots$ such that $\xi_1, \ldots, \xi_k$ are elements of T^* and $H_j = \langle \xi_1, \ldots, \xi_j \rangle$.

(i): Put $\xi_j = \chi_1 + \chi_{j+1}$.

(ii): Using the work of Fisher [20, 21], put $z_n = (q^n - 1)/(q-1) + 1$ and $x_n = z_n - n - 1$, for positive integers n. Let $G_1 = \langle \chi_1 \rangle$ and $G_n = \langle G_{n-1}, \chi_{z_{n-1}} \rangle$ for $n \geqslant 2$. To define $\xi_{x_n+1}, \xi_{x_n+2}, \ldots, \xi_{x_{n+1}}$, label the nonzero elements of G_n as η_i for $1 \leqslant i \leqslant q^n - 1 = x_{n+1} - x_n$. Then put

$$\xi_{x_n+i} = \eta_i + \chi_{z_n} + \chi_{z_n+i}$$

for $1 \leqslant i \leqslant x_{n+1} - x_n$.

(iii): This is similar to (ii), using the work of Bose [9, §5.4]. For $n \geqslant 2$ put $z_n = 2^{n-1} + 1$ and $x_n = z_n - n - 1$. Let $G_2 = \langle \chi_1, \chi_2 \rangle$ and $G_n = \langle G_{n-1}, \chi_{z_{n-1}} \rangle$ for $n \geqslant 3$. Label the nonzero elements of G_n for which $|J(\eta)|$ is even as η_i for $1 \leqslant i \leqslant 2^{n-1} - 1 = x_{n+1} - x_n$. Then define ξ_{x_n+i} as in (ii). ∎

This lemma may be true in other cases, but a general proof seems hard to find.

THEOREM 9. *Let q be a prime number; w, k, and e_k positive integers; $e_1, e_2, \ldots, e_{k-1}$ nonnegative integers; and*

$$\tilde{H} = \bigoplus_{j=1}^{k} (C_{q^j})^{e_j}.$$

Call T a $(w, \tilde{H})$-candidate if T is a name extension of

$$T' = \prod_{j=1}^{k} (q^j)^{F(w,j)},$$

where

$$F(w,k) = f_q(w, E_k),$$

$$F(w,j) = f_q(w, E_j) - f_q(w, E_{j+1})$$

for $j < k$, and $E_j = e_k + e_{k-1} + \cdots + e_j$. (In particular, $F(2, j) = e_j + \delta_{jk}$.) Then

(a) *T^* has a w-heavy subgroup H isomorphic to $\tilde{H}$ only if T is a $(w, \tilde{H})$-candidate;*

(b) *if T is a $(w, \tilde{H})$-candidate and either* (i) *$w \in \{2,3\}$ or* (ii) *$w = 4$ and $q = 2$ or* (iii) *$E_1 = 2$ or* (iv) *$k = 1$, then T^* has a w-heavy subgroup H isomorphic to $\tilde{H}$.*

Proof. (a): Let $j \in \{1, \ldots, k\}$. Then H contains a subgroup D isomorphic to $(C_{q^j})^{E_j}$. Let $L = \{q^{j-1}\chi : \chi \in D\}$. Then $L \cong (C_q)^{E_j}$, so Theorem 7 shows that T^* has a subgroup isomorphic to $(C_q)^{f_q(w, E_j)}$. Since each element of this subgroup has the form $q^{j-1}\chi$ for some χ in T^*, it follows that T is a name extension of $(q^j)^{f_q(w, E_j)}$. This is true for all j: hence the values of F given in the statement of the theorem.

(b): Put $y_j = f_q(w, E_j)$ for $j = 1, \ldots k$. Let $U = \oplus_{i=1}^{y_1} \langle u_i \rangle$, where each u_i has order q, and let η_i be a generator of $\langle u_i \rangle^*$. When any of conditions (i)–(iv) holds, the lemma shows that we can find a w-heavy subgroup K of U^* such that $K_j \cong (C_q)^{E_j}$ for $j = 1, \ldots, k$, where $K_j = \{\eta \in K : J(\eta) \subseteq \{1, \ldots, y_j\}\}$. Choose a minimal set D_j of generators for K_j such that $D_k \subseteq D_{k-1} \subseteq \cdots \subseteq D_1$. Thus $|D_k| = e_k$ and $|D_j \setminus D_{j+1}| = e_j$ for $1 \leqslant j < k$.

Put $m(i) = k$ for $1 \leqslant i \leqslant y_k$ and, for $1 \leqslant j \leqslant k - 1$, put $m(i) = j$ if $y_{j+1} < i \leqslant y_j$. Then T' has elements g_i $(1 \leqslant i \leqslant y_1)$ such that $T' = \oplus_{i=1}^{y_1} \langle g_i \rangle$ and g_i has order $q^{m(i)}$. Let χ_i be a generator of $\langle g_i \rangle^*$. For ξ in $D_j \setminus D_{j+1}$, if $\xi = \sum_i \xi_i \eta_i$ put $\chi(\xi) = \sum_i \xi_i q^{m(i)-j} \chi_i$: thus $\chi(\xi)$ is an element of T^* of order q^j. Let H' be the subgroup of T'^* generated by $\{\chi(\xi) : \xi \in D_1\}$; then H' is isomorphic to $\tilde{H}$. Consider the name homomorphism $\theta : T'^* \to U^*$ defined by $\theta(\chi_i) = \eta_i$. Since $\theta(H') = K$ and name homomorphisms do not increase weight, H' is w-heavy. Finally, if $\psi : T \to T'$ is a name epimorphism, then $\psi^*(H')$ is a w-heavy subgroup of T^* isomorphic to $\tilde{H}$. ■

EXAMPLE 10. Let $\tilde{H} = C_4 \oplus C_2$ and $w = 3$. When $q = 2$ we have $f_2(3,1) = 3$ and $f_2(3,2) = 5$, so $T' = (4)^3 \times (2)^2$. If $U = (2)^5$ we may take $K_2 = \langle A + B + C \rangle$, $K = K_1 = \langle A + B + C, A + D + E \rangle$; pulling A, B, and C back to 4-level factors (still called A, B, C), we obtain $H' = \langle A + B + C, 2A + D + E \rangle$, so that

$$H' = \{0, A + B + C, 2A + 2B + 2C, 3A + 3B + 3C, 2A + D + E,$$
$$3A + B + C + D + E, 2B + 2C + D + E, A + 3B + 3C + D + E\}.$$

If $T = (8) \times (4)^4$ with factors $\bar{A}, \ldots, \bar{E}$, then if ψ maps the generators of T to those of T' in order, we have $\psi^*(A) = 2\bar{A}$, $\psi^*(B) = \bar{B}$, $\psi^*(C) = \bar{C}$, $\psi^*(D) = 2\bar{D}$, $\psi^*(E) = 2\bar{E}$, so that

$$H = \psi^*(H') = \langle 2\bar{A} + \bar{B} + \bar{C}, 4\bar{A} + 2\bar{D} + 2\bar{E} \rangle.$$

A single replicate block design in 8 blocks with $L_{\text{blocks}} = H$ confounds the following effects:

$$[2\bar{A} + \bar{B} + \bar{C}],\quad [6\bar{A} + \bar{B} + \bar{C} + 2\bar{D} + 2\bar{E}] \qquad \text{(dimension 2 each)},$$

$$[4\bar{A} + 2\bar{B} + 2\bar{C}],\quad [4\bar{A} + 2\bar{D} + 2\bar{E}],\quad [2\bar{B} + 2\bar{C} + 2\bar{D} + 2\bar{E}]$$
$$\text{(dimension 1 each)}.$$

Subgroups H_1 and H_2 of T^* will be called *essentially the same* if there is a name isomorphism ψ of T such that $\psi^*(H_1) = H_2$. Even when $T = T'$ in Theorem 9, there may be essentially different choices of H. For example, $f_2(3,5) = 9$, and there are 5 essentially different 3-heavy subgroups of T^* isomorphic to $(C_2)^5$ when $T = (2)^9$. When $T \neq T'$ there are often name epimorphisms ψ_1 and ψ_2 from T to T' such that $\psi_1^*(H')$ and $\psi_2^*(H')$ are essentially different. In Example 10, because every name isomorphism of U which fixes K must fix $\langle A \rangle$, we obtain an essentially different subgroup of T^* by allowing ψ to transpose the first two generators: that is, interchange $2\bar{A}$ and $\bar{B}$ in H. For the same reason, there are two essentially different 3-heavy subgroups of T^* isomorphic to $C_2 \oplus C_2$ when $T = (2)^4 \times (4)$.

Theorems 5–9 show that, if pseudofactors are not used, all resolution-II single-replicate block designs, resolution-III fractions, and resolution-IV fractions may be obtained from such designs in which all factors have the same, prime number of levels. Designs with this last property will be called *elementary*, by analogy with the corresponding groups. Thus all the information in Dean and John's [16] list of 51 resolution-II block designs is contained in the 17 constituent 2-designs, 3 constituent 3-designs, and 1 constituent 5-design (not counting unblocked single replicates): moreover, 6 elementary 2-designs give all the information in the 17 2-designs. This way of summarizing the list, in addition to being an easy method of finding it, shows that it omits a $2\times4\times5$ design in 2 blocks of 20. Similarly, the 77 resolution-III fractional designs given by Lewis [31] are obtained from 23 2-designs, 3 3-designs, and other unblocked single replicates; no fewer than 12 designs listed are simply the combination of the unique resolution-III $\frac{1}{2}$-replicate of 2^3 with an unblocked single replicate. Again, 7 elementary 2-designs and 3 elementary 3-designs contain all the information in the list.

If the lemma is not true for $w \geqslant 5$, or for $w = 4$ when q is odd, then such dramatic simplifications may not be possible in general. However, Theorem 9 and its proof still give a good strategy for trying to construct designs from quotient elementary designs. Moreover, the first case not covered by Theorems 7 and 9(b) has $k = 2$ and $E_1 = 3$, and $w = 5$ (if $q = 2$) or $w = 4$ (if q is odd). For any given prime q, the corresponding group $\tilde{H}$ of smallest order is $C_{q^2}\oplus(C_q)^2$. Theorem 9(a) shows that if T has a w-heavy subgroup isomorphic to $\tilde{H}$, then $|T|$ is divisible by q^N, where $N = 2f_q(w,1)+[f_q(w,3)-f_q(w,1)] = f_q(w,3)+f_q(w,1) = f_q(w,3)+w$. Calculation shows that $f_2(5,3) = 10$, while the results of Bose [9, §§5.3, 5.5] show that $f_3(4,3) = 7$ and $f_q(4,3) = 6$ for $q \geqslant 5$. Hence $|T|$ is divisible by 2^{15}, 3^{11}, or q^{10} for $q \geqslant 5$. Thus Theorems 7 and 9(b) suffice for up to 32,767 treatments, which is enough for most practical purposes.

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