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Since we have not justified the previous steps mathematically, we must test the result numerically. Table 1 gives a comparison of the approximation (6) with the results of the numerical integration of Ruben (1954) for the cases $1/\rho = 1, 2, \ldots, 12$. [If $\rho = 0$, equation (6) is exact.]

The agreement in Table 1 is exceptionally good except when ρ is near unity. If greater accuracy is desired, it may be obtained by the use of *seven* terms of equation (4) and the application of more complicated non-linear transformations, as given by Shanks (1955).*

The approximation (6) is considerably more accurate than another one described elsewhere by the author (McFadden, 1955), which was based on an analogy with Pólya's urn scheme; yet the formula (6) is no more complicated than the other.

If the multivariate normal integral is known for four variables, the result for five follows immediately, as shown by David (1953).

Attempts to extend the present method to n variables, with all correlation coefficients equal, and also to the *general* quadrivariate case (with correlation coefficients unequal) have been unsuccessful.

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Weighted probits allowing for a non-zero response in the controls

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Black (1950) has shown how standard probit calculations can be carried out using tables of 'weighted probits'. These functions are defined by

$$\pi_0, \quad \text{weighted minimum probit} \equiv w(Y - P/Z),$$

$$\pi_1, \quad \text{weighted maximum probit} \equiv w(Y + Q/Z),$$

where Y is the probit value corresponding to a proportion P, Z is the corresponding ordinate, $w = Z^2/(PQ)$ is the probit weighting function and $Q = 1-P$. If r subjects are observed to respond and s not to respond, the appropriate weight is $rw + sw$ and the working probit multiplied by the weight is $r\pi_1 + s\pi_0$. Since r and s are usually fairly small integers these are simple computations, and the necessary table look-ups are more convenient than those required when using the tables of Finney (1952) or Fisher & Yates (1953).

Weighted probits have certain advantages when probit calculations are carried out on high-speed automatic computers. In desk computation, the working probit is usually found by adding a fraction of the range $(1/Z)$ to the minimum working probit $(Y - P/Z)$. Neither of these functions is particularly well behaved, each becoming numerically large for extreme values of Y, and Finney (1952) has provided an extensive table of double entry to by-pass this stage of the computation. This table is too large to be held in the store of present-day machines. By contrast, the weighted probit functions behave reasonably from a numerical point of view in the range of practical importance, and, since high-order interpolation is a fast and simple process on automatic computers, wide-interval tables with comparatively few entries can be used which can be accommodated in the store of the machine without difficulty. It is of course possible to avoid the use of tables altogether by computing the functions *ab initio* as required, but in the present instance the necessary routines would be rather complicated and would probably occupy almost as much room in the store as the tables. The process would also be more time-consuming than a simple table look-up. A further advantage in the use of tables is that the graduating function

* The author (J.A.M.) recommends the use of the iterated first-order transformation for each numerical value of ϕ.

can be changed from probits to, for example, logits or angles without any alteration in the main programme, simply by inserting the appropriate tables.

An extension of the simple probit technique is required when it is necessary to allow for a non-zero response rate untreated material. Finney (1952, pp. 88–91) has shown that when this 'natural response rate' is well determined and not too large, its effect can be allowed for by two simple modifications to the ordinary process. In the first place, the observed proportions responding, P, are adjusted by the so-called Abbot's correction to give adjusted proportions $P' = (P-C)/(1-C)$, where C is the natural response rate; secondly, the weights have to be multiplied by a factor $P'/\{P'+C/(1-C)\}$. It is of some interest that these alterations can be taken care of by modifying the tables of weights and weighted probits, thus enabling the original machine programme to be used unchanged.

It is somewhat simpler to work in terms of Normal Equivalent Deviates rather than probits, the addition of 5 being in any case irrelevant when working on automatic computers. The necessary alterations to the weights, w, are taken care of by inserting a suitably modified table. The working probit, y, is given by

$$y = Q'y_0 + P'y_1,$$

where P' is the adjusted proportion responding, $Q' = 1-P'$ and y_0, y_1 are the minimum and maximum working probits. In terms of the observed proportion responding, this gives

$$y = \frac{Q}{1-C}y_0 + \frac{P-C}{1-C}y_1,$$

so that the required value, nwy, is given in terms of the weighted probits by

$$\begin{aligned} nwy &= s\frac{\pi_0}{1-C} + r\frac{\pi_1}{1-C} - \frac{(r+s)\,C}{1-C}\pi_1 \\ &= s\left(\frac{\pi_0 - C\pi_1}{1-C}\right) + r\pi_1. \end{aligned}$$

Thus the form of the computation remains unaltered if we use the adjusted weighted probits

$$\pi_0' = (\pi_0 - C\pi_1)/(1-C)$$

and π_1, where the modified weights are used in computing π_0 and π_1 as functions of Y.

Tables of w, $\frac{1}{8}\pi_0'$ and $\frac{1}{8}\pi_1$ have been prepared covering the range of N.E.D.'s $-4(0\cdot125)+4$, with $C = 0(1)\,9\,\%$. Four-point interpolation in these tables (a convenient method is that described in Fisher & Yates (1953, p. 33)) gives 5-decimal accuracy over almost the whole range. Similar tables for the logit and angle function are in preparation.

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Treatment variances for experimental designs with serially correlated observations

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1. INTRODUCTION AND SUMMARY

Williams (1952) has considered the design of field experiments in which the fertilities of neighbouring plots are assumed to follow a linear autoregressive scheme of order one or two. Here a formal generalization is made and a notation introduced which simplifies the calculation of variances of estimates.

A model of the following form is assumed:

$$x_t + a_1x_{t-1} + a_2x_{t-2} + \dots + a_px_{t-p} = \epsilon_t, \tag{1}$$